On coupled Schrödinger systems with double critical exponents and indefinite weights*

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Abstract

By using variational methods, we study the existence of mountain pass solution to the following doubly critical Schrödinger system:

$$\begin{cases} -\Delta u - \mu_1 \frac{u}{|x|^2} - |u|^{2^*-2}u &= h(x)\alpha |u|^{\alpha-2}|v|^{\beta}u \quad \text{in } \mathbb{R}^N, \\ -\Delta v - \mu_2 \frac{v}{|x|^2} - |v|^{2^*-2}v &= h(x)\beta |u|^{\alpha}|v|^{\beta-2}v \quad \text{in } \mathbb{R}^N, \end{cases}$$

where $\alpha \geq 2, \beta \geq 2, \alpha+\beta \leq 2^*$; $\mu_1, \mu_2 \in [0, \frac{(N-2)^2}{4})$. The weight function $h(x)$ is allowed to be sign-changing so that the nonlinearities include a large class of indefinite weights. We show that the PS condition is satisfied at higher energy level when $\alpha + \beta = 2^*$ and obtain the existence of mountain pass solution. Besides, a nonexistence result of the ground state is given.

Key words: Doubly critical system, mountain pass solution, Nehari manifold, indefinite weight.

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1 Introduction

In this paper, we investigate the existence of solutions to the following nonlinear Schrödinger system:

$$\begin{cases} -\Delta u - \mu_1 \frac{u}{|x|^2} - |u|^{2^*-2}u &= h(x)\alpha|u|^{\alpha-2}|v|^{\beta}u, \ \text{ in } \mathbb{R}^N, \\ -\Delta v - \mu_2 \frac{v}{|x|^2} - |v|^{2^*-2}v &= h(x)\beta|u|^{\alpha}|v|^{\beta-2}v, \ \text{ in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\alpha \geq 2, \beta \geq 2, \alpha+\beta \leq 2^*$; $\mu_1, \mu_2 \in [0, \Lambda_N), \Lambda_N := \frac{(N-2)^2}{4}$; $h(x) \in L^\infty(\mathbb{R}^N)$. The interest for such systems is motivated by its applications to plasma physics, nonlinear optics, condensed matter physics, etc. For example, the coupled nonlinear Schrödinger systems arise in the description of several physical phenomena such as the propagation of pulses in birefringent optical fibers and Kerr-like photorefractive media, see [2, 18, 23, 24, 33, 14], etc. Also, it is related to the following Gross-Pitaevskii equations (cf. [17, 32]):

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 - a(x)\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \nu|\Phi_2|^2\Phi_1, \ x \in \mathbb{R}^N, \ t > 0, \\ -i\frac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 - b(x)\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \nu|\Phi_1|^2\Phi_2, \ x \in \mathbb{R}^N, \ t > 0, \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, \quad j = 1, 2, \\ \Phi_j(x,t) \to 0, \quad \text{as } |x| \to +\infty, \ t > 0, \ j = 1, 2, \end{cases} \tag{1.2}$$

where i is the imaginary unit; $a(x), b(x)$ are potential functions. Problem (1.2) also arises in the Hartree-Fock theory for a double condensate, i.e., a binary mixture of Bose-Einstein condensates in two different hyperfine states (see [13]).

We call a solution (u,v) nontrivial if both $u \not\equiv 0$ and $v \not\equiv 0$; we call a solution (u,v) semi-trivial if (u,v) is a type of $(u,0)$ or $(0,v)$. The existence of semi-trivial solution is equivalent to the solution of the following scalar equation:

$$-\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-2}u, \text{ in } \mathbb{R}^N, \tag{1.3}$$

whose solutions have been figured out. Here, when $\mu = 0$, we refer the readers to [4, 5, 6, 7, 8, 9, 15, 16]. When $\mu \in (0, \frac{(N-2)^2}{4})$, by [31], an explicit solution of (1.3) exists, namely

$$z_1^\mu(x) := \frac{A(N,\mu)}{|x|^{a_\mu}\left(1+|x|^{2-\frac{4a_\mu}{N-2}}\right)^{\frac{N-2}{2}}}; \tag{1.4}$$

where $a_\mu := \frac{N-2}{2} - \sqrt{(\frac{N-2}{2})^2 - \mu}$ and $A(N,\mu) := \left[\frac{N(N-2-2a_\mu)^2}{N-2}\right]$. This solution is also known to be the unique positive solution up to a conformal transformation of the form

$$z_\sigma^\mu = \sigma^{-\frac{N-2}{2}} z_1^\mu(\frac{x}{\sigma}), \quad \sigma > 0. \tag{1.5}$$

Before returning to the existence and nonexistence of the nontrivial solutions of (1.1), we recall the very recent paper [1], where the authors studied the existence of solutions to the following system:

$$\begin{cases} -\Delta u - \mu_1 \frac{u}{|x|^2} - |u|^{2^*-2}u & = \nu \cdot h(x)\alpha|u|^{\alpha-2}|v|^\beta u, \text{ in } \mathbb{R}^N, \\ -\Delta v - \mu_2 \frac{v}{|x|^2} - |v|^{2^*-2}v & = \nu \cdot h(x)\beta|u|^{\alpha}|v|^{\beta-2} v, \text{ in } \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $\mu_1, \mu_2 \in (0, \frac{(N-2)^2}{4})$ and the parameter ν serves as a regulator. Throughout the paper [1], $h(x)$ satisfies the following condition:

$$h(x) \geq 0, h(x) \not\equiv 0, h(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{1.7}$$

Let

$$S(\mu_i) := \left(1 - \frac{4\mu_i}{(N-2)^2}\right)^{\frac{N-1}{N}} S, \tag{1.8}$$

where S is the sharp constant of $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq S \left(\int_{\mathbb{R}^N} |u|^{2^*} \, dx\right)^{\frac{2}{2^*}} \tag{1.9}$$

and

$$S = \frac{N(N-2)}{4} |\mathbb{S}^N|^{\frac{2}{N}} = \frac{N(N-2)}{4} 2^{\frac{2}{N}} \pi^{1+\frac{1}{N}} \Gamma(\frac{N+1}{2})^{-\frac{2}{N}}. \tag{1.10}$$

In particular, $S = 3(\frac{\pi}{2})^{\frac{4}{3}}$ when $N = 3$; $S = \frac{4\sqrt{6}\pi}{3}$ when $N = 4$.

If the coupling terms are of subcritical case, i.e., $\alpha+\beta < 2^*$, when $\max\{\alpha,\beta\} < 2$ the authors of [1] prove that the least energy c satisfies

$$c < \frac{1}{N}\big(\min\{S(\mu_1), S(\mu_2)\}\big)^{\frac{N}{2}}$$

and obtain the existence of nontrivial ground state solution; when $\max\{\alpha,\beta\} = 2$, the similar results hold provided that the regulator ν is large enough. If $\min\{\alpha,\beta\} > 2$, the ground state energy is achieved by and only by semi-trivial solutions; if $\min\{\alpha,\beta\} = 2$, the similar results hold provided the regulator μ small enough. When $N = 3$ and $S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2) < S^{\frac{N}{2}}$, $\frac{\Lambda_N-\mu_1}{\Lambda_N-\mu_2} > \frac{1}{2}$, they obtain the existence of mountain pass solution provided that ν is sufficiently small.

For the critical case, that is, $\alpha+\beta = 2^*$, [1] assumed that $h(x)$ is a radial function satisfying

$$\begin{cases} h \in L^\infty(\mathbb{R}^N), h \geq 0, h \not\equiv 0, h \text{ is continuous in a neighborhood of } 0 \text{ and } \infty, \\ h(0) = \lim\limits_{|x|\to\infty} h(x) = 0. \end{cases}$$

Then they obtained that if $N \geq 5, \max\{\alpha,\beta\} < 2$, then (1.6) possesses a nontrivial ground state solution; if $N = 3, 4, \min\{\alpha,\beta\} \geq 2, \mu_2 \leq \mu_1 < \frac{(N-2)^2}{4}, \frac{\Lambda_N-\mu_1}{\Lambda_N-\mu_2} > 2^{-\frac{2}{N-1}}$, they obtain the existence of mountain pass solution provided that the regulator ν is small enough. But for the case of $\alpha+\beta = 2^*$ and $h(x)$ is not radial, they only obtain the existence of ground state solution for $\max\{\alpha,\beta\} < 2$ and ν small enough .

Note that if $1 < \alpha < 2, 1 < \beta < 2, \alpha+\beta = 2^*$, hence $2^* < 4$, which means that the results in [1] do not include the case of dimension $N = 3, 4$ if $h(x)$ is not radial. If $\mu_2 < \mu_1$ and $\beta < 2$, they also obtain the existence of ground state solution provided ν small enough. However, if $\mu_1 = \mu_2$ or $\min\{\alpha,\beta\} \geq 2$, whether there still exists a nontrivial solution remains open. At the end of [1], the authors also impose a list of complicated conditions on $h(x)$ and emphasize that if $h(x)$ has a fixed sign, by using the perturbation argument, they obtain the existence of nontrivial solutions provided that ν is small enough. In [1, Theorem 3.8], they considered the case of $\alpha+\beta < 2^*, \alpha \geq 2, \beta \geq 2$. Note that for this case there must hold $N = 3$. More important, in order to prove the Palais-Smale compactness condition, they need that

$$S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2) < S^{\frac{N}{2}}. \tag{1.11}$$

Under this hypothesis, they obtained a mountain pass solution provided that ν is small enough. We emphasize that (1.11) can not hold for many ranges of μ_1 and μ_2; for example: $\mu_1 = 0$ or $\mu_2 = 0$, or $\mu_1, \mu_2 > 0$ such that $\mu_1 + \mu_2 \leq \frac{1}{4}$.

Naturally, we concern the following questions which are still standing open before us.

1) Whether the role of the parameter ν is essential and can be dropped?

2) What happens if $h(x)$ is not radial?

3) What happens if $h(x)$ is sign-changing?

4) What happens if (1.11) is not true, i.e., $S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2) \geq S^{\frac{N}{2}}$?

5) When $\alpha + \beta = 2^*$, whether there exists a mountain pass solution to (1.1)?

The main purpose of the present paper is to study the existence of mountain pass solution when the ground state energy is only achieved by semi-trivial solution. We will always assume $h(x)$ is sign-changing and not necessary radial. Now it is the place to state our results in the current paper. We need one of the following two conditions:

($\mathbf{H_1}$) $h(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$;

($\mathbf{H_1'}$) $h(x)$ is continuous in $\mathbb{R}^N$, $h(0) \leq 0$, $\limsup\limits_{|x|\to\infty} h(x) \leq 0$. Moreover, we assume that $\gamma := \|h_-\|_\infty \max\{\alpha, \beta\} < 1$, where $h_- := \min\{h, 0\}$.

Further, we suppose that $h(x)$ satisfies the following integrable condition:

($\mathbf{H_2}$) $\int_{\mathbb{R}^N} h(x)|z_\sigma^{\mu_1}|^\alpha |z_\sigma^{\mu_2}|^\beta dx > 0$ for some $\sigma > 0$, where z_σ^μ is defined in (1.5).

Let us denote

$$\Theta := \begin{cases} \|h_+(x)\|_{L^{\frac{2^*}{2^*-\alpha-\beta}}(\mathbb{R}^N)} & \text{if } \alpha + \beta < 2^*, \\ \|h_+(x)\|_{L^\infty(\mathbb{R}^N)} & \text{if } \alpha + \beta = 2^*, \end{cases} \tag{1.12}$$

where $h_+ := \max\{h, 0\}$. Without loss of generality, throughout this paper we always assume $\mu_2 \leq \mu_1$.

1.1 Nonexistence of the Nontrivial Least Energy Solution

The first main result of the current paper concerns with the nonexistence of the ground state to (1.1) for all $N \geq 3$.

Theorem 1.1. *Assume that either $\beta \geq 2, \mu_2 < \mu_1$ or $\alpha \geq 2, \beta \geq 2, \mu_2 = \mu_1 = \mu$. Further, suppose that*

$$\begin{cases} h(x) \text{ satisfies } (H_1) \text{ if } \alpha + \beta < 2^*; \\ h(x) \text{ satisfies } (H_1') \text{ if } \alpha + \beta = 2^*. \end{cases}$$

Then there exists $\Theta_0 > 0$, depending on $N, \alpha, \beta, \mu_1, \mu_2$, such that if $\Theta \leq \Theta_0$, then the least energy of the system is exactly equal to $\frac{1}{N} S^{\frac{N}{2}}(\mu_1)$. Moreover, it is achieved by and only by

- $(\pm z_\sigma^{\mu_1}, 0)$ *if $\mu_2 < \mu_1$, where $\sigma > 0$;*

- $(\pm z^{\mu}_{\sigma}, 0)$ *and* $(0, \pm z^{\mu}_{\sigma})$ *if* $\mu_2 = \mu_1 = \mu \neq 0$;
- $(\pm z_{\sigma,x_i}, 0)$ *and* $(0, \pm z_{\sigma,x_i})$ *if* $\mu_2 = \mu_1 = 0$, *where*

$$z_{\sigma,x_i} = \sigma^{-\frac{N-2}{2}} z_1(\frac{x - x_i}{\sigma}), z_1(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{[1 + |x|^2]^{\frac{N-2}{2}}}, \quad \sigma > 0, x_i \in \mathbb{R}^N.$$

That is, problem (1.1) has no nontrivial least energy solution.

Remark 1.1. *In the above theorem, the constant* Θ_0 *has an explicit formula in terms of* $N, \alpha, \beta, \mu_1, \mu_2$. *To avoid tedious notations, we prefer to give them in Section 4. For the system (1.6), the authors of [1] had constructed similar results (see[1, Theorem 3.4]). But they required that* $\alpha, \beta \geq 2$ *and* $h(x) \geq 0$. *Here we improve the results of [1, Theorem 3.4] to the system (1.1). When* $\mu_1 \neq \mu_2$, *we only require* $\beta \geq 2$. *Moreover,* $h(x)$ *is allowed to be sign-changing in our case.*

1.2 Mountain pass solution: the case of $N = 3$.

In this case, $\Lambda_N = \frac{1}{4}$.

Theorem 1.2. *Assume* $N = 3, \alpha \geq 2, \beta \geq 2, \alpha + \beta < 2^*$ *and* $h(x)$ *satisfies* (H_1) *and* (H_2). *Furthermore, assume that* $\frac{1}{2} < \frac{1-4\mu_1}{1-4\mu_2}$ *and that*

$$\text{either} \quad S^{\frac{N}{2}}(\mu_2) + S^{\frac{N}{2}}(\mu_1) \leq S^{\frac{N}{2}} \quad \text{or} \quad 2\Big(\frac{S(\mu_1) + S(\mu_2)}{2}\Big)^{\frac{N}{2}} > S^{\frac{N}{2}}. \tag{1.13}$$

Assume further

$$\Theta \leq 10^{-4}\big[(2 - 8\mu_1)^{\frac{2}{3}} - \big(1 - 4\mu_2\big)^{\frac{2}{3}}\big], \tag{1.14}$$

then the problem (1.1) has a nontrivial weak solution (u_0, v_0) *with* $u_0 \geq 0, v_0 \geq 0, u_0v_0 \not\equiv 0$.

Remark 1.2. *If* $\mu_2 = \mu_1$, *then the alternatives of (1.13) hold true automatically.*

The next theorem is about the case of $N = 3$ and $\alpha + \beta = 2^*$, which means that the coupling terms are of critical.

Theorem 1.3. *Assume* $N = 3, \alpha \geq 2, \beta \geq 2, \alpha + \beta = 2^*$ *and* $h(x)$ *satisfies* (H_1') *and* (H_2). *Furthermore, assume that* $\mu_1 + \mu_2 \neq 0, \frac{1}{2} < \frac{1-4\mu_1}{1-4\mu_2}$ *such that either*

$$S^{\frac{N}{2}}(\mu_2) + S^{\frac{N}{2}}(\mu_1) \leq S^{\frac{N}{2}} \quad \text{or} \quad 2\Big(\frac{S(\mu_1) + S(\mu_2)}{2}\Big)^{\frac{N}{2}} > S^{\frac{N}{2}}. \tag{1.15}$$

Assume further

$$\Theta \leq \min\Big\{\frac{\mu_1 + \mu_2}{12}, 10^{-4}\big[(2 - 8\mu_1)^{\frac{2}{3}} - \big(1 - 4\mu_2\big)^{\frac{2}{3}}\big]\Big\}, \tag{1.16}$$

then the problem (1.1) has a nontrivial weak solution (u_0, v_0) *such that* $u_0 \geq 0, v_0 \geq 0, u_0v_0 \not\equiv 0$.

1.3 Mountain pass solution: the case of $N = 4$.

For the case of $N = 4$, we know $2^* = 4$ and $\Lambda_N = 1$. If $\alpha \geq 2, \beta \geq 2, \alpha + \beta = 2^*$, we must have $\alpha = \beta = 2$. Thus, (1.1) becomes a type of Bose-Einstein Condensates (BEC) equation in $\mathbb{R}^4$:

$$\begin{cases} -\Delta u - \mu_1 \frac{u}{|x|^2} & = u^3 + 2h(x)v^2u, \quad \text{in } \mathbb{R}^4, \\ -\Delta v - \mu_2 \frac{v}{|x|^2} & = v^3 + 2h(x)u^2v, \quad \text{in } \mathbb{R}^4, \\ u \geq 0, v \geq 0. \end{cases} \tag{1.17}$$

Note that both the cubic terms (u^3 and v^3) and the coupling terms (v^2u and u^2v) on the right-hand sides of (1.17) are of critical growth.

Theorem 1.4. *Assume (H_1') and (H_2). Suppose $\mu_1+\mu_2\neq 0, \frac{1}{2}<\left(\frac{1-\mu_1}{1-\mu_2}\right)^{\frac{3}{2}}$ such that*

$$\text{either}\quad S^{\frac{N}{2}}(\mu_2)+S^{\frac{N}{2}}(\mu_1)\leq S^{\frac{N}{2}}\quad\text{or}\quad 2\Big(\frac{S(\mu_1)+S(\mu_2)}{2}\Big)^{\frac{N}{2}}>S^{\frac{N}{2}}. \tag{1.18}$$

Assume further

$$\Theta\leq\frac{2-(1-\mu_1)^{\frac{3}{2}}-(1-\mu_2)^{\frac{3}{2}}}{16}, \tag{1.19}$$

then the problem (1.17) has a nontrivial weak solution (u_0,v_0) such that $u_0\geq 0, v_0\geq 0, u_0v_0\not\equiv 0$.

Remark 1.3. *Basically, the upper bounds of Θ in the above theorems are not sharp. However, in order to determine an unambiguous range of Θ, we prefer to give the explicit formulas for those constants. The optimal range of Θ is an interesting open question.*

Remark 1.4. *One of the main difficulties of studying this kind of problems is the failure of the (PS) condition due to the critical term $|u|^{2^*-2}u$ and the unbounded domain $\mathbb{R}^N$, especially for the couple terms $|u|^{\alpha-2}u|v|^\beta$ and $|u|^\alpha|v|^{\beta-2}v$ with $\alpha+\beta=2^*$. One has to overcome the difficulties on determining the compactness threshold. People usually study the case of $c<\frac{1}{N}S^{\frac{N}{2}}$. One of the main innovation of our present work is that we obtain a nontrivial solution with energy higher than $\frac{1}{N}S^{\frac{N}{2}}$. We also have to overcome the difficulties brought by the indefinite sign of the weight function $h(x)$, especially when $h(x)$ is not radial.*

2 Nehari Manifold

Let $\mathbb{D}:=D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$, where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^N)}:=\Big(\int_{\mathbb{R}^N}|\nabla u(x)|^2dx\Big)^{1/2}.$$

By the Hardy inequality, when $0<\mu<\frac{(N-2)^2}{4}$, $\|u\|_{D^{1,2}(\mathbb{R}^N)}$ is equivalent to the following norm:

$$\|u\|_\mu:=\Big(\int_{\mathbb{R}^N}(|\nabla u(x)|^2-\mu\frac{u^2}{|x|^2})dx\Big)^{1/2}.$$

For simplicity, we will also use the notation of $\|u\|_0$ to represent $\|u\|_{D^{1,2}(\mathbb{R}^N)}$. For $(u,v)\in\mathbb{D}$, define the norm

$$\|(u,v)\|_{\mathbb{D}}=\Big(\|u\|_{\mu_1}^2+\|v\|_{\mu_2}^2\Big)^{1/2}.$$

A pair of function (u,v) is said to be a weak solution of problem (1.1) iff

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla u\cdot\nabla\varphi_1 dx-\mu_1\int_{\mathbb{R}^N}\frac{u\varphi_1}{|x|^2}dx+\int_{\mathbb{R}^N}\nabla v\cdot\nabla\varphi_2 dx-\mu_2\int_{\mathbb{R}^N}\frac{u\varphi_2}{|x|^2}dx\\
&-\int_{\mathbb{R}^N}|u|^{2^*}u\varphi_1 dx-\int_{\mathbb{R}^N}|v|^{2^*-2}v\varphi_2 dx-\alpha\int_{\mathbb{R}^N}h(x)|u|^{\alpha-2}u|v|^{\beta}\varphi_1 dx\\
&-\beta\int_{\mathbb{R}^N}h(x)|u|^{\alpha}|v|^{\beta-2}v\varphi_2 dx=0\quad\text{for all }(\varphi_1,\varphi_2)\in\mathbb{D}. \qquad (2.1)
\end{aligned}$$

Thus, the corresponding energy functional of problem (1.1) is defined by

$$\Phi(u,v)=\frac{1}{2}||(u,v)||_{\mathbb{D}}^2-\frac{1}{2^*}\Big(||u||_{L^{2^*}(\mathbb{R}^N)}^{2^*}+||v||_{L^{2^*}(\mathbb{R}^N)}^{2^*}\Big)-\int_{\mathbb{R}^N}h(x)|u|^{\alpha}|v|^{\beta}dx \qquad (2.2)$$

for all $(u,v)\in\mathbb{D}$. The associated Nehari manifold is defined as

$$\mathcal{N}:=\Big\{(u,v)\in\mathbb{D}\backslash\big\{(0,0)\big\}:J(u,v)=0\Big\},$$

where

$$\begin{aligned}
J(u,v) &= \Big\langle\Phi'(u,v),(u,v)\Big\rangle=||(u,v)||_{\mathbb{D}}^2-\Big(||u||_{L^{2^*}(\mathbb{R}^N)}^{2^*}\\
&\quad +||v||_{L^{2^*}(\mathbb{R}^N)}^{2^*}\Big)-(\alpha+\beta)\int_{\mathbb{R}^N}h(x)|u|^{\alpha}|v|^{\beta}dx, \qquad (2.3)
\end{aligned}$$

and $\Phi'(u,v)$ denotes the Fréchet derivative of Φ at (u,v), $\langle\cdot,\cdot\rangle$ is the duality product between $\mathbb{D}$ and its dual space $\mathbb{D}^*$. We have the following properties on the Nehari manifold.

Lemma 2.1. *Assume $\alpha+\beta\le 2^*$. In particular, if $\alpha+\beta=2^*$, we require (H_1') instead of (H_1). Then $\forall(u,v)\in\mathbb{D}\backslash\big\{(0,0)\big\}$, there exists a unique $t=t_{(u,v)}>0$ such that $t(u,v)=(tu,tv)\in\mathcal{N}$. Furthermore, there exists $\delta>0$ such that $t_{(u,v)}\ge\delta$ for all $(u,v)\in\mathcal{S}:=\Big\{(u,v)\in\mathbb{D}:||(u,v)||_{\mathbb{D}}^2=1\Big\}$, and $\mathcal{N}$ is closed and bounded away from $(0,0)$.*

Proof. For $(u,v)\in\mathbb{D}\backslash\big\{(0,0)\big\}$, we denote that

$$\begin{cases}a:=||u||_{L^{2^*}(\mathbb{R}^N)}^{2^*}+||v||_{L^{2^*}(\mathbb{R}^N)}^{2^*}>0;\\ b:=(\alpha+\beta)\int_{\mathbb{R}^N}h(x)|u|^{\alpha}|v|^{\beta}dx;\\ c:=||(u,v)||_{\mathbb{D}}^2>0.\end{cases} \qquad (2.4)$$

Then, $\frac{d}{dt}\Phi(tu,tv)=-tg(t)$, where $g(t):=a\ t^{2^*-2}+b\ t^{\alpha+\beta-2}-c$. Firstly we consider the case $b\ge 0$. Note that there exists a unique $t_0>0$ such that $g(t_0)=0$. Moreover, $g(t)<0$ for $0<t<t_0$ and $g(t)>0$ for $t>t_0$. Secondly we consider the case $b<0$. If $\alpha+\beta<2^*$, there exists some $s>0$ such that $g(s)=0$. Let t_0 be the minimum of the solutions of $g(t)=0$, that is, $g(t_0)=0$ and $g(t)<0$ for $t<t_0$. For $\forall\ t>t_0$, it is easy to check that $g'(t)>0$. Thus,

$g(t)>0$ for $t>t_0$. Then, t_0 is the unique solution to $g(t)=0$. If $\alpha+\beta=2^*$, by the Young's inequality and (H_1'), we have

$$\begin{aligned}
|b| &\leq \Big|2^*\int_{\mathbb{R}^N} h_-(x)|u|^\alpha|v|^\beta dx\Big| \\
&\leq 2^*\|h_-\|_\infty\big(\frac{\alpha}{2^*}||u||^{2^*}_{L^{2^*}(\mathbb{R}^N)}+\frac{\beta}{2^*}||v||^{2^*}_{L^{2^*}(\mathbb{R}^N)}\big) \\
&\leq \|h_-\|_\infty \max\{\alpha,\beta\}a \\
&< a. \qquad (2.5)
\end{aligned}$$

Hence, $a+b>0$. Then there exists a unique positive solution t_0 to the equation: $g(t)=(a+b)t^{2^*-2}-c=0$. In particular, $g(t)<0$ for $0<t<t_0$ and $g(t)>0$ for $t>t_0$. Let $t_{(u,v)}:=t_0$ be defined as above, we finally obtain that

$$\frac{d}{dt}\Phi(tu,tv)=-tg(t)\begin{cases}>0 & \text{for } 0<t<t_{(u,v)};\\ <0 & \text{for } t>t_{(u,v)}.\end{cases}$$

In either case, there exists a unique $t_{(u,v)}>0$ such that $\Phi\Big(t_{(u,v)}u,t_{(u,v)}v\Big)=\max\limits_{t>0}\Phi(tu,tv)$ and $t_{(u,v)}(u,v)\in\mathcal{N}$. For $\omega=(u,v)\in\mathcal{S}$, since $h(x)\in L^{\frac{N}{2}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and $\alpha+\beta\leq 2^*$, there exists some $C>0$ such that

$$\begin{cases}
a &= ||u||^{2^*}_{L^{2^*}(\mathbb{R}^N)}+||v||^{2^*}_{L^{2^*}(\mathbb{R}^N)}\leq C;\\
|b| &= \big|(\alpha+\beta)\int_{\mathbb{R}^N}h(x)|u|^\alpha|v|^\beta dx\big|\leq C;\\
c &= 1.
\end{cases}$$

We consider the equation $g(t)=a\ t^{2^*-2}+b\ t^{\alpha+\beta-2}-1=0$. If $b\leq 0$, we have $at^{2^*-2}\geq 1$, hence $t\geq (a)^{-\frac{1}{2^*-2}}\geq (C)^{-\frac{1}{2^*-2}}$. If $b>0$, then we have either $at^{2^*-2}\geq\frac{1}{2}$ or $bt^{\alpha+\beta-2}\geq\frac{1}{2}$. Thus, either $t\geq\Big(\frac{1}{2C}\Big)^{\frac{1}{2^*-2}}$ or $t\geq\Big(\frac{1}{2C}\Big)^{\frac{1}{\alpha+\beta-2}}$. Therefore, there exists $\delta>0$ such that $t_{(u,v)}\geq\delta$ for all $(u,v)\in S$. Therefore, $\mathcal{N}$ is bounded away from $(0,0)$. Obviously, $\mathcal{N}$ is closed. $\square$

3 Analysis of the Palais-Smale Sequences

In this section, we perform a careful analysis of the behavior of the Palais-Smale sequences with the aid of the concentration-compactness principle in [19, 20], which allows to recover the compactness below some critical threshold. Set

$$\widetilde{C}_{N,\alpha,\beta}:=\frac{\Big(1-\frac{4\max\{\mu_1,\mu_2\}}{(N-2)^2}\Big)^{\frac{1-N}{N}}-1}{\max\{\alpha,\beta\}}. \qquad (3.1)$$

Lemma 3.1. *Assume $\mu_2 = 0, \mu_1 = \mu \in (0, \frac{(N-2)^2}{4})$ and*

$$\text{either} \begin{cases} (H_1) \\ \alpha + \beta < 2^* \end{cases} \text{or} \begin{cases} (H_1') \\ \alpha + \beta = 2^* \end{cases}.$$

Let $\{(u_n, v_n)\} \subset \mathcal{N}$ be a Palais-Smale sequence for $\Phi|_{\mathcal{N}}$ at level $c < \frac{1}{N} S^{\frac{N}{2}}(\mu)$. Then, there exists some constant C, such that $||(u_n, v_n)||_{\mathbb{D}} \leq C$ for all $n \in \mathbb{N}$ and $\Phi'(u_n, v_n) \to 0$ in the dual space $\mathbb{D}^*$. Moreover,

(1) for the case of $\begin{cases} (H_1) \\ \alpha + \beta < 2^* \end{cases}$, we have $(u_n, v_n) \to (u_0, v_0)$ in $\mathbb{D}$ up to a subsequence;

(2) for the case of $\begin{cases} (H_1') \\ \alpha + \beta = 2^* \end{cases}$, if $h(x)$ is radial, we have $(u_n, v_n) \to (u_0, v_0)$ in $\mathbb{D}$ up to a subsequence. However, if $h(x)$ is not radial, we obtain the same result provided that the additional hypothesis $\Theta < \widetilde{C}_{N,\alpha,\beta}$ (see (3.1)) holds.

Proof. The ideas for proving this lemma are quite similar to the cases of $\mu_1, \mu_2 > 0$ and $\mu_1 = \mu_2 = 0$ in [35, Lemma 6.3], we omit the details. □

In the next section, we will study the nonexistence of nontrivial ground state solutions. In view of the nonexistence of the ground state to the system, we will investigate the existence of mountain pass solutions of system (1.1). For this goal, we need an improved Palais-Smail condition at higher energy level. Let

$$I_\mu(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx, \quad u \in D^{1,2}(\mathbb{R}^N). \tag{3.2}$$

We consider the following modified problem to find the nonnegative mountain pass solutions to (1.1),

$$\begin{cases} -\Delta u - \mu_1 \frac{u}{|x|^2} = u_+^{2^*-1} + \alpha h(x) u_+^{\alpha-1} v_+^\beta, & \text{in } \mathbb{R}^N, \\ -\Delta v - \mu_2 \frac{v}{|x|^2} = v_+^{2^*-1} + \beta h(x) u_+^\alpha v_+^{\beta-1}, & \text{in } \mathbb{R}^N, \end{cases} \tag{3.3}$$

where $u_+ = \max\{u, 0\}$. The weak solutions to problem (3.3) are critical points of the following functional $\overline{\Phi} : \mathbb{D} \to \mathbb{R}$ given by

$$\overline{\Phi}(u, v) = \overline{I}_{\mu_1}(u) + \overline{I}_{\mu_2}(v) - \int_{\mathbb{R}^N} h(x) u_+^\alpha v_+^\beta dx, \tag{3.4}$$

where

$$\overline{I}_{\mu_i}(w) = \frac{1}{2} Q_{\mu_i}(w) - \frac{1}{2^*} \int_{\mathbb{R}^N} w_+^{2^*} dx, \quad i = 1, 2, \tag{3.5}$$

and

$$Q_\mu(w) = \int_{\mathbb{R}^N} |\nabla w|^2 dx - \mu \int_{\mathbb{R}^N} \frac{w^2}{|x|^2} dx. \tag{3.6}$$

Obviously, the critical points of $\overline{\Phi}$ provide nonnegative solutions to the original problem (1.1). We denote by $\overline{\mathcal{N}}$ the Nehari manifold associated to $\overline{\Phi}$, i.e.,

$$\overline{\mathcal{N}} = \Big\{(u,v) \in \mathbb{D}\backslash\{(0,0)\} : \langle \overline{\Phi}'(u,v),(u,v)\rangle = 0\Big\}. \tag{3.7}$$

Assume $N=3, \frac{1}{2} < \frac{1-4\mu_1}{1-4\mu_2}, \Theta \leq \min\{C_1, C_2\}$, where Θ is defined in (1.12) and

$$C_1 := \frac{(1-4\mu_1)^{\frac{2}{3}} - \left(\frac{1}{2}\right)^{\frac{2}{3}}(1-4\mu_2)^{\frac{2}{3}}}{\left(\frac{1}{2}\right)^{\frac{\alpha-2}{6}}(1-4\mu_2)^{\frac{\alpha-2+4\beta}{6}}\alpha S^{\frac{\alpha-2}{4}+\beta-1}}, \tag{3.8}$$

$$C_2 := \frac{2-\sqrt[3]{2}}{2^{\frac{8-\beta}{6}}(1-4\mu_2)^{\frac{4\alpha+\beta-6}{6}}\beta S^{\frac{\beta-2}{4}+\alpha-1}}. \tag{3.9}$$

It is easy to check that $C_2 > 5\times 10^{-4}, C_1 > 10^{-3}\big[(1-4\mu_1)^{\frac{2}{3}} - \big(\dfrac{1}{2}\big)^{\frac{2}{3}}(1-4\mu_2)^{\frac{2}{3}}\big]$.

Lemma 3.2. *Assume $N=3$, $\alpha,\beta \geq 2$, $\alpha+\beta < 2^*$ and $\frac{1}{2} < \frac{1-4\mu_1}{1-4\mu_2}$. Let $\{(u_n,v_n)\} \subset \overline{\mathcal{N}}$ be a Palais-Smale sequence for $\overline{\Phi}|_{\overline{\mathcal{N}}}$ at level $c \in \mathbb{R}$. Then, there exists $C>0$ such that $||(u_n,v_n)||_{\mathbb{D}} \leq C$ for all $n \in \mathbb{N}$ and $\overline{\Phi}'(u_n,v_n) \to 0$ in the dual space $\mathbb{D}^*$. Furthermore, if $\Theta \leq \min\{C_1,C_2\}$ and c satisfies*

$$\frac{1}{N}S^{\frac{N}{2}}(\mu_2) < c < \frac{1}{N}S^{\frac{N}{2}}(\mu_2) + \inf_{(u,v)\in\overline{\mathcal{N}}} \Phi(u,v); \tag{3.10}$$

$$c \neq \frac{l}{N}S^{\frac{N}{2}}(\mu_1) \ \ \textit{and} \ \ c \neq \frac{l}{N}S^{\frac{N}{2}} \ \textit{for all} \ l \in \mathbb{N}\backslash\{0\}, \tag{3.11}$$

then up to a subsequence, $(u_n,v_n) \to (u_0,v_0)$ in $\mathbb{D}$.

.

Remark 3.1. *In [1, Lemma 3.5], the authors only considered the case $\frac{1}{N}S^{\frac{N}{2}}(\mu_1) + \frac{1}{N}S^{\frac{N}{2}}(\mu_2) < \frac{1}{N}S^{\frac{N}{2}}$. However, if $\mu_1+\mu_2 \leq \frac{1}{4}$, then $\frac{1}{N}S^{\frac{N}{2}}(\mu_1) + \frac{1}{N}S^{\frac{N}{2}}(\mu_2) < \frac{1}{N}S^{\frac{N}{2}}$ will never meet. In particular, the sign-changing $h(x)$ makes the proof in Lemma 3.2 more complicated.*

Proof. We divide the proof into five steps.
Step 1: It is easy to show that $\{(u_n,v_n)\}$ is bounded in $\mathbb{D}$ and $\overline{\Phi}'(u_n,v_n) \to 0$ in the dual space $\mathbb{D}^*$. Up to a subsequence, $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ converges weakly to some (u_0,v_0). Hence, $((u_n)_-,(v_n)_-) \to (0,0)$ strongly in $\mathbb{D}$. It follows that $((u_n)_+,(v_n)_+)$ is a bounded Palais-Smale sequence of $\overline{\Phi}$. For $((u_n)_+,(v_n)_+)$, we may find a $t_n>0$ such that $t_n((u_n)_+,(v_n)_+) \in \mathcal{N}\cap\overline{\mathcal{N}}$. Since $(u_n,v_n) \in$

$\overline{\mathcal{N}}$ and $((u_n)_-, (v_n)_-) \to 0$ in $\mathbb{D}$, we have $t_n \to 1$. Hence without loss of generality, we can assume that $u_n \geq 0, v_n \geq 0, \{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \mathcal{N} \cap \overline{\mathcal{N}}$ is a Palais-Smale sequence for $\overline{\Phi}$ at level c. Notice that $\Phi(u_n, v_n) = \overline{\Phi}(u_n, v_n)$. For the simplicity, we use $\|u\|_0$ to stand for $\|u\|_{D^{1,2}(\mathbb{R}^N)}$. There exists $(u_0, v_0) \in \mathbb{D}$ and a subsequence, still denoted as $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ such that

$$(u_n, v_n) \rightharpoonup (u_0, v_0) \quad \text{weakly in } \mathbb{D}, \quad (u_n, v_n) \to (u_0, v_0) \quad \text{a.e. in } \mathbb{R}^N, \tag{3.12}$$

$$\text{and} \quad (u_n, v_n) \to (u_0, v_0) \quad \text{strongly in } L^\gamma_{loc}(\mathbb{R}^N) \times L^\gamma_{loc}(\mathbb{R}^N) \text{ for all } \gamma \in [1, 2^*). \tag{3.13}$$

In view of the concentration-compactness principle due to Lions [19, 20], there exists a subsequence, still denoted as $\{(u_n, v_n)\}_{n\in\mathbb{N}}$, two at most countable sets $\mathcal{J}$ and $\mathcal{K}$, set of points $\{x_j \in \mathbb{R}^N\backslash\{0\} : j \in \mathcal{J}\}$ and $\{y_k \in \mathbb{R}^N\backslash\{0\} : k \in \mathcal{K}\}$, real numbers $\zeta_j, \rho_j, j \in \mathcal{J}, \overline{\zeta}_k, \overline{\rho}_k, k \in \mathcal{K}, \zeta_0, \rho_0, \overline{\zeta}_0$ and $\overline{\rho}_0$ such that

$$\begin{cases} |\nabla u_n|^2 \rightharpoonup d\mu \geq |\nabla u_0|^2 + \sum\limits_{j\in\mathcal{J}} \zeta_j \delta_{x_j} + \zeta_0\delta_0, \\ |\nabla v_n|^2 \rightharpoonup d\overline{\mu} \geq |\nabla v_0|^2 + \sum\limits_{k\in\mathcal{K}} \overline{\zeta}_k \delta_{y_k} + \overline{\zeta}_0\delta_0, \\ |u_n|^{2^*} \rightharpoonup d\rho = |u_0|^{2^*} + \sum\limits_{j\in\mathcal{J}} \rho_j \delta_{x_j} + \rho_0\delta_0, \\ |v_n|^{2^*} \rightharpoonup d\overline{\rho} = |v_0|^{2^*} + \sum\limits_{k\in\mathcal{K}} \overline{\rho}_k \delta_{y_k} + \overline{\rho}_0\delta_0, \\ \frac{u_n^2}{|x|^2} \rightharpoonup d\theta = \frac{u_0^2}{|x|^2} + \theta_0\delta_0, \\ \frac{v_n^2}{|x|^2} \rightharpoonup d\overline{\theta} = \frac{v_0^2}{|x|^2} + \overline{\theta}_0\delta_0. \end{cases} \tag{3.14}$$

Define

$$\begin{aligned} \zeta_\infty &:= \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |\nabla u_n|^2 dx, \ \rho_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |u_n|^{2^*} dx, \\ \overline{\zeta}_\infty &:= \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |\nabla v_n|^2 dx, \ \overline{\rho}_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |v_n|^{2^*} dx. \end{aligned} \tag{3.15}$$

It follows that

$$\zeta_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |\nabla (u_n - u_0)|^2 dx, \tag{3.16}$$

$$\rho_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |u_n - u_0|^{2^*} dx,$$

$$\overline{\zeta}_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |\nabla (v_n - v_0)|^2 dx, \tag{3.17}$$

$$\overline{\rho}_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |v_n - v_0|^{2^*} dx. \tag{3.18}$$

From the Sobolev's inequality, it follows easily that

$$S\rho_j^{\frac{2}{2^*}} \leq \zeta_j \quad \text{for all } j \in \mathcal{J}; \qquad S\overline{\rho}_k^{\frac{2}{2^*}} \leq \overline{\zeta}_k \quad \text{for all } k \in \mathcal{K}. \tag{3.19}$$

Step 2: We prove that either $u_n \to u_0$ strongly in $L^{2^*}(\mathbb{R}^N)$ or $v_n \to v_0$ strongly in $L^{2^*}(\mathbb{R}^N)$. If not, then there exist some $j_0 \in \mathcal{J} \cup \{0, \infty\}$ and $k_0 \in \mathcal{K} \cup \{0, \infty\}$ such that $\rho_{j_0} > 0, \overline{\rho}_{k_0} > 0$. Since $\alpha + \beta < 2^*$, we have $\rho_0 \geq S^{\frac{N}{2}}(\mu_1), \overline{\rho}_{k_0} \geq S^{\frac{N}{2}}(\mu_2)$. In order to make the present paper easy to follow, we prefer to give part of the proofs. Indeed, for $\varepsilon > 0$, let ϕ_j^ε be a smooth cut-off function centered at x_j, $0 \leq \phi_j^\varepsilon \leq 1$, such that

$$\phi_j^\varepsilon(x) = \begin{cases} 1 & \text{if } |x - x_j| \leq \frac{\varepsilon}{2} \\ 0 & \text{if } |x - x_j| \geq \varepsilon \end{cases} \quad \text{and } |\nabla_y \phi_j^\varepsilon(y + x_j)| \leq \frac{4}{\varepsilon} \text{ for all } y = x - x_j \in \mathbb{R}^N. \tag{3.20}$$

Testing $\Phi'(u_n, v_n)$ with $(u_n\phi_j^\varepsilon, 0)$, we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\langle \Phi'(u_n, v_n), (u_n\phi_j^\varepsilon, 0) \right\rangle \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \Big(|\nabla u_n|^2 \phi_j^\varepsilon + u_n \nabla u_n \cdot \nabla \phi_j^\varepsilon - \phi_j^\varepsilon |u_n|^{2^*} \\ &\quad -\alpha h(x)|u_n|^\alpha |v_n|^\beta \phi_j^\varepsilon \Big) dx. \end{aligned} \tag{3.21}$$

Notice that for all $\varepsilon > 0$ fixed,

$$\begin{aligned} \int_{\mathbb{R}^N} u_n \nabla u_n \cdot \nabla \phi_j^\varepsilon dx &= \int_{\frac{\varepsilon}{2} \leq |x - x_j| \leq \varepsilon} u_n \nabla u_n \cdot \nabla \phi_j^\varepsilon dx \\ &= \int_{\frac{\varepsilon}{2} \leq |x - x_j| \leq \varepsilon} (u_n - u_0) \nabla u_n \cdot \nabla \phi_j^\varepsilon dx \\ &\quad + \int_{\frac{\varepsilon}{2} \leq |x - x_j| \leq \varepsilon} u_0 \nabla u_n \cdot \nabla \phi_j^\varepsilon dx \\ &:= I + II. \end{aligned} \tag{3.22}$$

Note $|\nabla \phi_j^\varepsilon| < \frac{4}{\varepsilon}$. Without loss of generality, we may assume that

$$(u_n, v_n) \to (u_0, v_0) \quad \text{strongly in } L^\gamma_{loc}(\mathbb{R}^N) \times L^\gamma_{loc}(\mathbb{R}^N) \text{ for all } \gamma \in [1, 2^*). \tag{3.23}$$

Then by (3.23) and the boundedness of ∇u_n in $L^2(\mathbb{R}^N)$, we have

$$I \to 0 \quad \text{as } n \to +\infty. \tag{3.24}$$

Note that II can be taken as a linear functional in $L^2(\mathbb{R}^N)$ and that $u_n \rightharpoonup u_0$ in $D^{1,2}(\mathbb{R}^N)$, we have

$$II \to \int_{\frac{\varepsilon}{2} \leq |x - x_j| \leq \varepsilon} u_0 \nabla u_0 \cdot \nabla \phi_j^\varepsilon dx \quad \text{as } n \to +\infty. \tag{3.25}$$

Since $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in $\mathbb{D}$, (u_0, v_0) is a weakly solution to problem (1.1). Taking $(u_0\phi_j^\varepsilon, 0)$ as the testing function, then we have

$$0 = \int_{\mathbb{R}^N} \left[|\nabla u_0|^2 \phi_j^\varepsilon + u_0 \nabla u_0 \cdot \nabla \phi_j^\varepsilon - \phi_j^\varepsilon |u_0|^{2^*} - \alpha h(x) |u_0|^\alpha |v_0|^\beta \phi_j^\varepsilon \right] dx. \tag{3.26}$$

Further,

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n|^{\alpha}|v_n|^{\beta}\phi_j^{\varepsilon}dx &= \int_{|x-x_j|\le\varepsilon} |u_n|^{\alpha}|v_n|^{\beta}\phi_j^{\varepsilon}dx \\ &\le \int_{|x-x_j|\le\varepsilon} |u_n|^{\alpha}|v_n|^{\beta}dx \\ &\le \big(\int_{|x-x_j|\le\varepsilon} |u_n|^{2^*}dx\big)^{\frac{\alpha}{2^*}}\big(\int_{|x-x_j|\le\varepsilon} |v_n|^{2^*}dx\big)^{\frac{\beta}{2^*}} \\ &\quad \big(\int_{|x-x_j|\le\varepsilon} 1dx\big)^{\frac{2^*-\alpha-\beta}{2^*}} \\ &= O(\varepsilon^{\frac{N(2^*-\alpha-\beta)}{2^*}}).\end{aligned}$$

Since $h(x)\in L^{\infty}(\mathbb{R}^N)$, we have that

$$\int_{\mathbb{R}^N} \alpha h(x)|u_n|^{\alpha}|v_n|^{\beta}\phi_j^{\varepsilon}dx = O(\varepsilon^{\frac{N(2^*-\alpha-\beta)}{2^*}}). \tag{3.27}$$

Especially,

$$\int_{\mathbb{R}^N} \alpha h(x)|u_0|^{\alpha}|v_0|^{\beta}\phi_j^{\varepsilon}dx = O(\varepsilon^{\frac{N(2^*-\alpha-\beta)}{2^*}}). \tag{3.28}$$

By (3.14) and (3.21)$\sim$(3.28), let $\varepsilon\to 0$ we obtain that

$$\mu_j - \rho_j \le 0. \tag{3.29}$$

By (3.19), we conclude that for all $j\in\mathcal{J}$, either $\rho_j = 0$ or $\rho_j \ge S^{\frac{N}{2}}$, which also implies that $\mathcal{J}$ is finite. For the details about the similar results related to $j\in\{0,\infty\}$ and $\overline{\rho}_k, k\in\mathcal{K}\cup\{0,\infty\}$, we refer the readers to [35, Lemma 3.2]. Then we have

$$\begin{aligned}c = \Phi(u_n,v_n)+o(1) &\ge \big(\frac{1}{2}-\frac{1}{\alpha+\beta}\big)\big(S(\mu_1)\rho_{j_0}^{\frac{2}{2^*}} + S(\mu_2)\overline{\rho}_{k_0}^{\frac{2}{2^*}}\big) \\ &\quad +\big(\frac{1}{\alpha+\beta}-\frac{1}{2^*}\big)(\rho_{j_0}+\overline{\rho}_{k_0}) \\ &\ge \frac{1}{N}\big(S^{\frac{N}{2}}(\mu_1)+S^{\frac{N}{2}}(\mu_2)\big),\end{aligned}$$

which is a contradiction with (3.10).

Step 3: We prove that either $u_n\to u_0$ strongly in $D^{1,2}(\mathbb{R}^N)$ or $v_n\to v_0$ strongly in $D^{1,2}(\mathbb{R}^N)$. By Step 2, if $u_n\to u_0$ strongly in $L^{2^*}(\mathbb{R}^N)$, then

$$\|u_n-u_0\|_{\mu_1}^2 = \langle\Phi'(u_n,v_n),(u_n-u_0,0)\rangle + o(1) = o(1) \text{ as } n\to+\infty.$$

Hence, $u_n\to u_0$ strongly in $D^{1,2}(\mathbb{R}^N)$ in this case. If $v_n\to v_0$ strongly in $L^{2^*}(\mathbb{R}^N)$, correspondingly we have $v_n\to v_0$ strongly in $D^{1,2}(\mathbb{R}^N)$.

Step 4: If $v_n \to v_0$ strongly in $D^{1,2}(\mathbb{R}^N)$, we prove $u_n \to u_0$ strongly in $D^{1,2}(\mathbb{R}^N)$. We argue by contradiction and assume that $u_n \rightharpoonup u_0$ weakly but none of its subsequence converges strongly to u_0.

Firstly we claim $v_0 \not\equiv 0$. If not, $v_0 \equiv 0$, it is easy to check that $\{u_n\}$ is a nonnegative Palais-Smale sequence for the functional I_{μ_1} defined in (3.2), at the energy level $c = \lim\limits_{n\to\infty} I_{\mu_1}(u_n)$, which can be calculated as following:

$$
\begin{aligned}
c = &\lim_{n\to\infty} \Phi(u_n, v_n)\\
= &\lim_{n\to\infty} \frac{1}{2}\|(u_n, v_n)\|_{\mathbb{D}}^2 - \frac{1}{2^*}\Big(|u_n|_{L^{2^*}(\mathbb{R}^N)}^{2^*} + |v_n|_{L^{2^*}(\mathbb{R}^N)}^{2^*}\Big) - \int_{\mathbb{R}^N} h(x)|u_n|^\alpha |v_n|^\beta dx\\
= &\lim_{n\to\infty} I_{\mu_1}(u_n) + I_{\mu_2}(v_n) - \int_{\mathbb{R}^N} h(x)|u_n|^\alpha |v_n|^\beta dx\\
= &\lim_{n\to\infty} I_{\mu_1}(u_n) \text{ since } v_n \to v_0 \equiv 0.
\end{aligned}
$$

Then the result of [26, Theorem 3.1] (we take $K(x) \equiv 1, \lambda = \mu_1$ in [26]) implies that there exists some $m, l \in \mathbb{N}$ such that

$$
c = \lim_{n\to\infty} \Phi(u_n, v_n) = \lim_{n\to\infty} I_{\mu_1}(u_n) = I_{\mu_1}(u_0) + \frac{m}{N} S^{\frac{N}{2}} + \frac{l}{N} S^{\frac{N}{2}}(\mu_1).
$$

If $u_0 \equiv 0$, then by (3.11), we have $m \neq 0, l \neq 0$. In this case, $c \geq \frac{1}{N} S^{\frac{N}{2}} + \frac{1}{N} S^{\frac{N}{2}}(\mu_1)$, a contradiction with (3.10). If $u_0 \not\equiv 0$, since $u_0 \geq 0$, we have $u_0 = z_\sigma^{\mu_1}$ for some $\sigma > 0$ and $\int_{\mathbb{R}^N} |u_0|^{2^*} dx = S^{\frac{N}{2}}(\mu_1), I_{\mu_1}(u_0) = \frac{1}{N} S^{\frac{N}{2}}(\mu_1)$. By (3.11), we obtain $m \neq 0$. Then $c \geq \frac{1}{N} S^{\frac{N}{2}} + \frac{1}{N} S^{\frac{N}{2}}(\mu_1)$, also a contradiction with (3.10). Thereby the claim $v_0 \not\equiv 0$ is proved.

Thus we may assume that $u_n \rightharpoonup u_0$ weakly but not strongly in $D^{1,2}(\mathbb{R}^N)$ and $v_n \to v_0 \not\equiv 0$ strongly in $D^{1,2}(\mathbb{R}^N)$.If $u_0 \equiv 0$, then v_0 weakly solves

$$
-\Delta v_0 - \mu_2 \frac{v_0}{|x|^2} - v_0^{2^*-1} = 0,
$$

By the known result in Section 1, we have $v_0 = z_\sigma^{\mu_2}$ which is defined in (1.5) for

some $\sigma > 0$. Thus, $|v_0|^{2^*}_{L^{2^*}(\mathbb{R}^N)} = \|v_0\|^2_{\mu_2} = S^{\frac{N}{2}}(\mu_2)$. Therefore,

$$
\begin{aligned}
c =& \big(\frac{1}{2} - \frac{1}{\alpha+\beta}\big)\|(u_n, v_n)\|^2_{\mathbb{D}} + \big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\big)\big(|u_n|^{2^*}_{L^{2^*}(\mathbb{R}^N)} + |v_n|^{2^*}_{L^{2^*}(\mathbb{R}^N)}\big) + o(1) \\
\geq& \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\Big[\|(u_0, v_0)\|^2_{\mathbb{D}} + \sum_{j\in\mathcal{J}\cup\{0,\infty\}} \zeta_j + \sum_{k\in\mathcal{K}\cup\{0,\infty\}} \overline{\zeta}_k\Big] + \\
& \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)\Big[\int_{\mathbb{R}^N}(|u_0|^{2^*} + |v_0|^{2^*})dx + \sum_{j\in\mathcal{J}\cup\{0,\infty\}} \rho_j + \sum_{k\in\mathcal{K}\cup\{0,\infty\}} \overline{\rho}_k\Big] + o(1) \\
=& \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\|v_0\|^2_{\mu_2} + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)|v_0|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \\
& + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)\big[\sum_{j\in\mathcal{J}\cup\{0,\infty\}} \zeta_j\big] + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)\big[\sum_{j\in\mathcal{J}\cup\{0,\infty\}} \rho_j\big] + o(1) \\
\geq& \frac{1}{N}|v_0|^{2^*}_{L^{2^*}(\mathbb{R}^N)} + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)\big[\sum_{j\in\mathcal{J}} S\rho_j^{\frac{2}{2^*}} + S(\mu_1)\rho_0^{\frac{2}{2^*}} + S(\mu_1)\rho_\infty^{\frac{2}{2^*}}\big] \\
& + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)\big[\sum_{j\in\mathcal{J}} \rho_j + \rho_0 + \rho_\infty\big] + o(1).
\end{aligned}
$$

Since $u_n \rightharpoonup u_0$ weakly but not strongly in $D^{1,2}(\mathbb{R}^N)$, there exists some $j \in \mathcal{J} \cup \{0, \infty\}$ such that $\rho_j \neq 0$ and that

$$
\begin{aligned}
c \geq& \frac{1}{N}|v_0|^{2^*}_{L^{2^*}(\mathbb{R}^N)} + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)S(\mu_1)\rho_j^{\frac{2}{2^*}} + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)\rho_j \\
\geq& \frac{1}{N}|v_0|^{2^*}_{L^{2^*}(\mathbb{R}^N)} + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)S(\mu_1) \cdot S^{\frac{N}{2}\cdot\frac{2}{2^*}}(\mu_1) + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)S^{\frac{N}{2}}(\mu_1) \\
=& \frac{1}{N}\big(S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2)\big),
\end{aligned}
$$

a contradiction with (3.10). Hence we can assume that $u_0 \not\equiv 0$. It is clear that $(u_0, v_0) \in \mathcal{N} \cap \overline{\mathcal{N}}$ and

$$
\Phi(u_0, v_0) = \frac{1}{N}\Big(\int_{\mathbb{R}^N} u_0^{2^*}dx + \int_{\mathbb{R}^N} v_0^{2^*}dx\Big) + \frac{\alpha+\beta-2}{2}\int_{\mathbb{R}^N} h(x)u_0^\alpha v_0^\beta dx. \quad (3.30)
$$

Since $v_n \to v_0$ in $D^{1,2}(\mathbb{R}^N)$ and $u_n \rightharpoonup u_0$ weakly in $D^{1,2}(\mathbb{R}^N)$, we can obtain that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x)u_n^\alpha v_n^\beta dx = \int_{\mathbb{R}^N} h(x)u_0^\alpha v_0^\beta dx. \quad (3.31)
$$

Combining these facts, we have

$$
\begin{aligned}
\Phi(u_n, v_n) =& \frac{1}{2}\|(u_n, v_n)\|^2_{\mathbb{D}} - \frac{1}{2^*}\Big(|u_n|^{2^*}_{L^{2^*}(\mathbb{R}^N)} + |v_n|^{2^*}_{L^{2^*}(\mathbb{R}^N)}\Big) - \int_{\mathbb{R}^N} h(x)|u_n|^\alpha|v_n|^\beta dx \\
=& \frac{1}{N}\int_{\mathbb{R}^N} u_n^{2^*}dx + \frac{1}{N}\int_{\mathbb{R}^N} v_n^{2^*}dx + \frac{\alpha+\beta-2}{2}\int_{\mathbb{R}^N} h(x)u_n^\alpha v_n^\beta dx,
\end{aligned}
\quad (3.32)
$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} v_n^{2^*}dx=\int_{\mathbb{R}^N} v_0^{2^*}dx, \tag{3.33}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} u_n^{2^*}dx=\int_{\mathbb{R}^N} u_0^{2^*}dx+\sum_{j\in\mathcal{J}}\rho_j+\rho_0+\rho_\infty. \tag{3.34}$$

By (3.30) $\sim$ (3.34), we obtain that

$$\begin{aligned}\Phi(u_0,v_0) &= \lim_{n\to\infty}\Phi(u_n,v_n)-\frac{1}{N}\big(\sum_{j\in\mathcal{J}}\rho_j+\rho_0+\rho_\infty\big)\\ &= c-\frac{1}{N}\big(\sum_{j\in\mathcal{J}}\rho_j+\rho_0+\rho_\infty\big)\\ &\le c-\frac{1}{N}S^{\frac{N}{2}}(\mu_1)\\ &< \frac{1}{N}S^{\frac{N}{2}}(\mu_2).\end{aligned} \tag{3.35}$$

Note that

$$\int_{\mathbb{R}^N} u_0^{2^*}dx+\alpha\int_{\mathbb{R}^N} h(x)u_0^{\alpha}v_0^{\beta}dx=\|u_0\|_{\mu_1}^2\ge S(\mu_1)\big(\int_{\mathbb{R}^N} u_0^{2^*}dx\big)^{\frac{2}{2^*}}. \tag{3.36}$$

If $\int_{\mathbb{R}^N} h(x)u_0^{\alpha}v_0^{\beta}dx\le 0$, then $\int_{\mathbb{R}^N} u_0^{2^*}dx\ge S^{\frac{N}{2}}(\mu_1)$. Similarly, we have $\int_{\mathbb{R}^N} v_0^{2^*}dx\ge S^{\frac{N}{2}}(\mu_2)$. Thus,

$$\begin{aligned}\Phi(u_0,v_0) &= \big(\frac{1}{2}-\frac{1}{\alpha+\beta}\big)\|(u_0,v_0)\|_{\mathbb{D}}^2+\big(\frac{1}{\alpha+\beta}-\frac{1}{2^*}\big)\big(\|u_0\|_{L^{2^*}}^{2^*}+\|v_0\|_{L^{2^*}}^{2^*}\big)\\ &\ge \big(\frac{1}{2}-\frac{1}{\alpha+\beta}\big)S(\mu_1)\|u_0\|_{L^{2^*}}^2+\big(\frac{1}{2}-\frac{1}{\alpha+\beta}\big)S(\mu_2)\|v_0\|_{L^{2^*}}^2\\ &\quad+\big(\frac{1}{\alpha+\beta}-\frac{1}{2^*}\big)\big(\|u_0\|_{L^{2^*}}^{2^*}+\|v_0\|_{L^{2^*}}^{2^*}\big)\\ &\ge \big(\frac{1}{2}-\frac{1}{\alpha+\beta}\big)S(\mu_1)S^{\frac{N}{2^*}}(\mu_1)+\big(\frac{1}{\alpha+\beta}-\frac{1}{2^*}\big)S^{\frac{N}{2}}(\mu_1)\\ &\quad+\big(\frac{1}{2}-\frac{1}{\alpha+\beta}\big)S(\mu_2)S^{\frac{N}{2^*}}(\mu_2)+\big(\frac{1}{\alpha+\beta}-\frac{1}{2^*}\big)S^{\frac{N}{2}}(\mu_2)\\ &= \frac{1}{N}\big(S^{\frac{N}{2}}(\mu_1)+S^{\frac{N}{2}}(\mu_2)\big)>\frac{1}{N}S^{\frac{N}{2}}(\mu_2),\end{aligned} \tag{3.37}$$

a contradiction with (3.35).

If $\int_{\mathbb{R}^N} h(x)u_0^{\alpha}v_0^{\beta}dx>0$, then by (3.30) and (3.35) we deduce that

$$\|u_0\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}+\|v_0\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}<S^{\frac{N}{2}}(\mu_2). \tag{3.38}$$

Similar to (3.36), we have

$$\int_{\mathbb{R}^N} v_0^{2^*}dx+\beta\int_{\mathbb{R}^N} h(x)u_0^{\alpha}v_0^{\beta}dx=\|v_0\|_{\mu_2}^2\ge S(\mu_2)\big(\int_{\mathbb{R}^N} v_0^{2^*}dx\big)^{\frac{2}{2^*}}. \tag{3.39}$$

Then by (3.38), (3.36) and (3.39) we obtain that

$$S(\mu_2)\|v_0\|^2_{L^{2^*}(\mathbb{R}^N)} \le \|v_0\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} + \beta\Theta S^{\frac{N-2}{4}\alpha}(\mu_2)\|v_0\|^{\beta}_{L^{2^*}(\mathbb{R}^N)} \tag{3.40}$$

and

$$S(\mu_1)\|u_0\|^2_{L^{2^*}(\mathbb{R}^N)} \le \|u_0\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} + \alpha\Theta S^{\frac{N-2}{4}\beta}(\mu_2)\|u_0\|^{\alpha}_{L^{2^*}(\mathbb{R}^N)}. \tag{3.41}$$

Since $N=3$, we have $S(\mu)=(1-4\mu)^{\frac{2}{3}}S$, then (3.40) and (3.41) are equivalent to

$$\big(1-4\mu_2\big)^{\frac{2}{3}}S \le \|v_0\|^4_{L^6(\mathbb{R}^3)} + \beta\Theta\big(1-4\mu_2\big)^{\frac{2\alpha}{3}}S^{\alpha}\|v_0\|^{\beta-2}_{L^6(\mathbb{R}^3)} \tag{3.42}$$

and

$$\big(1-4\mu_1\big)^{\frac{2}{3}}S \le \|u_0\|^4_{L^6(\mathbb{R}^3)} + \alpha\Theta\big(1-4\mu_2\big)^{\frac{2\beta}{3}}S^{\beta}\|u_0\|^{\alpha-2}_{L^6(\mathbb{R}^3)}. \tag{3.43}$$

Note that $\alpha\ge 2$ and that $f(t):=t^{\frac{2}{3}}+\alpha\Theta(1-4\mu_2)^{\frac{2}{3}\beta}S^{\beta}t^{\frac{\alpha-2}{6}}$ is increasing in $(0,+\infty)$. If $\Theta\le C_1$, where C_1 is given in (3.8), then

$$\begin{aligned}
& f\Big(\frac{1}{2}(1-4\mu_2)S^{\frac{3}{2}}\Big)\\
&= \Big[\frac{1}{2}(1-4\mu_2)S^{\frac{3}{2}}\Big]^{\frac{2}{3}} + \alpha\Theta(1-4\mu_2)^{\frac{2}{3}\beta}S^{\beta}\Big[\frac{1}{2}(1-4\mu_2)S^{\frac{3}{2}}\Big]^{\frac{\alpha-2}{6}}\\
&= \Big(\frac{1}{2}\Big)^{\frac{2}{3}}\big(1-4\mu_2\big)^{\frac{2}{3}}S + \Big(\frac{1}{2}\Big)^{\frac{\alpha-2}{6}}\alpha\Theta\big(1-4\mu_2\big)^{\frac{2}{3}\beta}\big(1-4\mu_2\big)^{\frac{\alpha-2}{6}}S^{\beta+\frac{\alpha-2}{4}}\\
&\le \big(1-4\mu_1\big)^{\frac{2}{3}}S.
\end{aligned}$$

Recalling (3.43), it follows that $\|u_0\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \ge \frac{1}{2}(1-4\mu_2)S^{\frac{3}{2}} = \frac{1}{2}S^{\frac{N}{2}}(\mu_2)$. Similarly, we have $\|v_0\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \ge \frac{1}{2}S^{\frac{N}{2}}(\mu_2)$. Thus, $\|u_0\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} + \|v_0\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \ge S^{\frac{N}{2}}(\mu_2)$, a contradiction to (3.38).

Step 5: If $u_n\to u_0$ strongly in $D^{1,2}(\mathbb{R}^N)$, we show that v_n converges strongly. For this case, $\mathcal{K}\cup\{0,\infty\}$ is reduced to be at most one point. In fact, if not, by (3.31), (3.39), we obtain that $c\ge \frac{2}{N}S^{\frac{N}{2}}(\mu_2)\ge \frac{1}{N}\big(S^{\frac{N}{2}}(\mu_1)+S^{\frac{N}{2}}(\mu_2)\big)$, contradicting to (3.10). Assume that $v_n\rightharpoonup v_0$ weakly but none of its subsequence converges strongly to v_0. We claim $u_0\not\equiv 0$. If not, $u_0\equiv 0$ and $v_0\not\equiv 0$, then $v_0\in D^{1,2}(\mathbb{R}^N)$ is a weak solution to

$$-\Delta v_0 - \mu_2\frac{v_0}{|x|^2} = |v_0|^{2^*-1}v_0. \tag{3.44}$$

Denote

$$E_1 := \inf\big\{I_{\mu_2}(v)\big|v\neq 0 \text{ is a solution of problem (3.44)}\big\},$$

$$E_2 := \inf\Big\{\int_{\mathbb{R}^N}|v|^{2^*}dx\Big|v\neq 0 \text{ is a solution of problem (3.44)}\Big\},$$

where $I_\mu(v)$ is defined in (3.2). Then it is well known that $E_1 = \frac{1}{N} S^{\frac{N}{2}}(\mu_2), E_2 = S^{\frac{N}{2}}(\mu_2)$, and they are only achieved by

$$\begin{cases} \pm z_\sigma^{\mu_2}, \sigma > 0 \ \text{ if } \mu_2 > 0 \\ \pm z_{\sigma, x_i}, \sigma > 0, x_i \in \mathbb{R}^N \text{ if } \mu_2 = 0. \end{cases}$$

Thus, by the fact that $\{v_n\}$ concentrate at exactly one point, we obtain that

$$c \geq \frac{1}{N} \Big(\int_{\mathbb{R}^N} v_0^{2^*} dx + S^{\frac{N}{2}}(\mu_2) \Big) \geq \frac{2}{N} S^{\frac{N}{2}}(\mu_2) \geq \frac{1}{N} (S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2)),$$

contradicting (3.10). On the other hand, if $u_0 \equiv 0, v_0 \equiv 0$, then v_n solves

$$-\Delta v_n - \mu_2 \frac{v_n}{|x|^2} - v_n^{2^*-1} = o(1), \quad \text{in the dual space} \big(D^{1,2}(\mathbb{R}^N) \big)^*. \tag{3.45}$$

Then an analogous argument as that in step 4 will lead to that

$$\begin{aligned} c &= \Phi(u_n, v_n) + o(1) = I_{\mu_2}(v_n) + o(1) \to I_{\mu_2}(v_0) + \frac{m}{N} S^{\frac{N}{2}}(\mu_2) + \frac{l}{N} S^{\frac{N}{2}} \\ &= \frac{m}{N} S^{\frac{N}{2}}(\mu_2) + \frac{l}{N} S^{\frac{N}{2}} \text{ for some } m, l \in \mathbb{N} \end{aligned}$$

as $n \to \infty$. By (3.11), we have $m \neq 0, l \neq 0$, then $c \geq \frac{1}{N} S^{\frac{N}{2}}(\mu_2) + \frac{1}{N} S^{\frac{N}{2}}$, a contradiction with (3.10). Thereby the claim $u_0 \not\equiv 0$ is proved. Thus we may assume that $u_n \to u_0 \not\equiv 0$ strongly in $D^{1,2}(\mathbb{R}^N)$, and $v_n \rightharpoonup v_0$. If $v_0 \equiv 0$, then u_0 weakly solves

$$-\Delta u_0 - \mu_1 \frac{u_0}{|x|^2} - u_0^{2^*-1} = 0,$$

hence $u_0 = z_\sigma^{\mu_1}$ for some $\sigma > 0$ and $\displaystyle\int_{\mathbb{R}^N} |u_0|^{2^*} dx = S^{\frac{N}{2}}(\mu_1)$. When $\mu_2 > 0$ (similarly for the case of $\mu_2 = 0$), by the fact that $\{v_n\}$ concentrates to exactly

one point, we deduce that

$$\begin{aligned}
c &= \Big(\frac{1}{2}-\frac{1}{\alpha+\beta}\Big)\Big[\|u_0\|^2_{\mu_1}+\|v_0\|^2_{\mu_2} \\
&\quad +\sum_{k\in\mathcal{K}}\overline{\zeta}_k+(\overline{\zeta}_0-\mu_2\overline{\theta}_0)+(\overline{\zeta}_\infty-\mu_2\overline{\theta}_\infty)\Big]+\Big(\frac{1}{\alpha+\beta}-\frac{1}{2^*}\Big) \\
&\quad \Big[\int_{\mathbb{R}^N}(|u_0|^{2^*}+|v_0|^{2^*})dx+\sum_{k\in\mathcal{K}}\overline{\rho}_k+\overline{\rho}_0+\overline{\rho}_\infty\Big]+o(1) \\
&\geq \Big(\frac{1}{2}-\frac{1}{\alpha+\beta}\Big)\Big[\|u_0\|^2_{\mu_1}+S\sum_{k\in\mathcal{K}}\overline{\rho}_k^{\frac{2}{2^*}}+S(\mu_2)(\overline{\rho}_0^{\frac{2}{2^*}}+\overline{\rho}_\infty^{\frac{2}{2^*}})\Big] \\
&\quad +\Big(\frac{1}{\alpha+\beta}-\frac{1}{2^*}\Big)\Big(\|u_0\|^{2^*}_{2^*}+\sum_{k\in\mathcal{K}}\overline{\rho}_k+\overline{\rho}_0+\overline{\rho}_\infty\Big)+o(1) \\
&\geq \Big(\frac{1}{2}-\frac{1}{\alpha+\beta}\Big)\Big(S(\mu_1)\|u_0\|^2_{2^*}+S(\mu_2)\overline{\rho}_{\widetilde{k}}^{\frac{2}{2^*}}\Big) \\
&\quad +\Big(\frac{1}{\alpha+\beta}-\frac{1}{2^*}\Big)\Big(\|u_0\|^{2^*}_{2^*}+\overline{\rho}_{\widetilde{k}}\Big)\quad \text{for some } \widetilde{k}\in\mathcal{K}\cup\{0,\infty\}+o(1) \\
&\geq \frac{1}{N}\big(S^{\frac{N}{2}}(\mu_1)+S^{\frac{N}{2}}(\mu_2)\big)+o(1), \qquad (3.46)
\end{aligned}$$

a contradiction with (3.10). Hence, both $u_0\not\equiv 0$ and $v_0\not\equiv 0$. Hence,

$$\begin{aligned}
c &= \Phi(u_n,v_n)-\frac{1}{2}\big\langle\Phi'(u_n,v_n),(u_n,v_n)\big\rangle+o(1) \qquad (3.47)\\
&= \frac{1}{N}\Big(\int_{\mathbb{R}^N}u_n^{2^*}dx+\int_{\mathbb{R}^N}v_n^{2^*}dx\Big)+\frac{\alpha+\beta-2}{2}\int_{\mathbb{R}^N}h(x)u_n^\alpha v_n^\beta dx+o(1),
\end{aligned}$$

it follows that, for some $k\in\mathcal{K}\cup\{0,\infty\}$,

$$c=\frac{1}{N}\big(\int_{\mathbb{R}^N}u_0^{2^*}dx+\int_{\mathbb{R}^N}v_0^{2^*}dx+\overline{\rho}_k\big)+\frac{\alpha+\beta-2}{2}\int_{\mathbb{R}^N}h(x)u_0^\alpha v_0^\beta dx. \qquad (3.48)$$

On the other hand, recall that $\langle\Phi'(u_n,v_n),(u_0,v_0)\rangle=o(1)$, we have

$$\|(u_0,v_0)\|^2_{\mathbb{D}}=\int_{\mathbb{R}^N}u_0^{2^*}dx+\int_{\mathbb{R}^N}v_0^{2^*}dx+(\alpha+\beta)\int_{\mathbb{R}^N}h(x)u_0^\alpha v_0^\beta dx,$$

i.e., $(u_0,v_0)\in\mathcal{N}$. If $\mu_2>0$, by (3.31), (3.48) and assumption (3.10), we obtain that

$$\begin{aligned}
\Phi(u_0,v_0) &= \frac{1}{N}\Big(\int_{\mathbb{R}^N}u_0^{2^*}dx+\int_{\mathbb{R}^N}v_0^{2^*}dx\Big)+\frac{\alpha+\beta-2}{2}\int_{\mathbb{R}^N}h(x)u_0^\alpha v_0^\beta dx \\
&= c-\frac{\overline{\rho}_k}{N}<\frac{1}{N}S^{\frac{N}{2}}(\mu_2)+\inf_{(u,v)\in\mathcal{N}}\Phi(u,v)-\frac{\overline{\rho}_k}{N} \\
&\leq \frac{1}{N}S^{\frac{N}{2}}(\mu_2)+\inf_{(u,v)\in\mathcal{N}}\Phi(u,v)-\frac{1}{N}S^{\frac{N}{2}}(\mu_2) \\
&= \inf_{(u,v)\in\mathcal{N}}\Phi(u,v), \qquad (3.49)
\end{aligned}$$

a contradiction with $(u_0,v_0)\in\mathcal{N}$. $\square$

Next, we consider the case of $\alpha+\beta=2^*$. For this case, we can not expect $\rho_j=0$ or $\rho_j\geq S^{\frac{N}{2}}$ for $j\notin\{0,\infty\}$ any more. Because the Hölder's inequality is not enough to ensure

$$\int_{\mathbb{R}^N} h(x)|u_n|^\alpha|v_n|^\beta\phi_j^\varepsilon dx\to 0 \text{ uniformly as } \varepsilon\to 0, \tag{3.50}$$

where ϕ_j^ε is defined in (3.20). Thus, the step 1 of Lemma 3.2 fails for the case of $\alpha+\beta=2^*$. We will impose more conditions on $h(x)$ to overcome this difficulty. Assume $\mu_1+\mu_2\neq 0$, take ε_1 small enough such that

$$2(1-\varepsilon_1)^{\frac{N-2}{N}}>\big(1-\frac{4\mu_1}{(N-2)^2}\big)^{\frac{N-1}{2}}+\big(1-\frac{4\mu_2}{(N-2)^2}\big)^{\frac{N-1}{2}}. \tag{3.51}$$

For example, if $N=3$, we choose ε_1 satisfying

$$\varepsilon_1<1-\big[1-2(\mu_1+\mu_2)\big]^3 \tag{3.52}$$

and if $N=4$, we may take ε_1 satisfying

$$\varepsilon_1<1-\frac{\big[(1-\mu_1)^{\frac{3}{2}}+(1-\mu_2)^{\frac{3}{2}}\big]^2}{4}. \tag{3.53}$$

Lemma 3.3. *Assume $\alpha+\beta=2^*$, $0\leq\mu_2\leq\mu_1<\frac{1}{4}, 1-4\mu_2<2-8\mu_1,\alpha,\beta\geq 2,\mu_1+\mu_2\neq 0$. Let $\{(u_n,v_n)\}\subset\overline{\mathcal{N}}$ be a Palais-Smale sequence for $\overline{\Phi}|_{\overline{\mathcal{N}}}$ at level $c\in\mathbb{R}$. Then, there exists some constant C such that $||(u_n,v_n)||_{\mathbb{D}}\leq C$ for all $n\in\mathbb{N}$ and $\overline{\Phi}'(u_n,v_n)\to 0$ in the dual space $\mathbb{D}^*$. Moreover, if c satisfies (3.10), (3.11) and*

$$\Theta\leq\min\Big\{\frac{1-(1-\varepsilon_1)^{\frac{2}{N}}}{2^{\frac{\beta}{2}}\alpha(1-\varepsilon_1)^{\frac{\alpha-2}{2^*}}},\frac{1-(1-\varepsilon_1)^{\frac{2}{N}}}{2^{\frac{\alpha}{2}}\beta(1-\varepsilon_1)^{\frac{\beta-2}{2^*}}},C_1,C_2\Big\}, \tag{3.54}$$

where ε_1 satisfies (3.52) and C_1,C_2 are defined in (3.8) and (3.9), then, up to a subsequence, $(u_n,v_n)\to(u_0,v_0)$ in $\mathbb{D}$.

Remark 3.2. *Under the assumptions of Lemma 3.3, we can have*

$$\min\Big\{\frac{1-(1-\varepsilon_1)^{\frac{2}{N}}}{2^{\frac{\beta}{2}}\alpha(1-\varepsilon_1)^{\frac{\alpha-2}{2^*}}},\frac{1-(1-\varepsilon_1)^{\frac{2}{N}}}{2^{\frac{\alpha}{2}}\beta(1-\varepsilon_1)^{\frac{\beta-2}{2^*}}},C_1,C_2\Big\}>$$
$$\min\Big\{\frac{\mu_1+\mu_2-(\mu_1+\mu_2)^2}{6},10^{-3}\big[(1-4\mu_1)^{\frac{2}{3}}-\big(\frac{1}{2}\big)^{\frac{2}{3}}\big(1-4\mu_2\big)^{\frac{2}{3}}\big],5\times 10^{-4}\Big\}.$$

Proof. We need several steps.
Step 1: There exist an at most countable set $\mathcal{J}$ (for simplicity, here we view $\mathcal{J}$ as the set $\mathcal{J}\cup\mathcal{K}$ in Lemma 3.2), the set of points $\big\{x_j\in\mathbb{R}^N\backslash\{0\}:j\in J\big\}$,

real numbers $\zeta_j, \rho_j, \overline{\zeta}_j, \overline{\rho}_j, j \in J, \zeta_0, \rho_0, \overline{\zeta}_0$ and $\overline{\rho}_0$, such that

$$\begin{cases} |\nabla u_n|^2 \rightharpoonup d\mu \geq |\nabla u_0|^2 + \sum\limits_{j\in\mathcal{J}} \zeta_j \delta_{x_j} + \zeta_0\delta_0, \\ |\nabla v_n|^2 \rightharpoonup d\overline{\mu} \geq |\nabla v_0|^2 + \sum\limits_{j\in\mathcal{J}} \overline{\zeta}_j \delta_{x_j} + \overline{\zeta}_0\delta_0, \\ |u_n|^{2^*} \rightharpoonup d\rho = |u_0|^{2^*} + \sum\limits_{j\in\mathcal{J}} \rho_j \delta_{x_j} + \rho_0\delta_0, \\ |v_n|^{2^*} \rightharpoonup d\overline{\rho} = |v_0|^{2^*} + \sum\limits_{j\in\mathcal{J}} \overline{\rho}_j \delta_{x_j} + \overline{\rho}_0\delta_0, \\ \frac{u_n^2}{|x|^2} \rightharpoonup d\theta = \frac{u_0^2}{|x|^2} + \theta_0\delta_0, \\ \frac{v_n^2}{|x|^2} \rightharpoonup d\overline{\theta} = \frac{v_0^2}{|x|^2} + \overline{\theta}_0\delta_0. \end{cases} \tag{3.55}$$

Note $\Phi'(u_n, v_n) \to 0$ in D^* and $\{(u_n, v_n)\}$ is bounded, we obtain that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \big\langle \Phi'(u_n, v_n), (u_n\phi_j^\varepsilon, 0)\big\rangle \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \Big(|\nabla u_n|^2\phi_j^\varepsilon + u_n\nabla u_n \cdot \nabla\phi_j^\varepsilon - \mu_1 \frac{u_n^2\phi_j^\varepsilon}{|x|^2} \\ &\quad -\phi_j^\varepsilon|u_n|^{2^*} - \alpha h(x)|u_n|^\alpha|v_n|^\beta\phi_j^\varepsilon\Big)dx. \end{aligned} \tag{3.56}$$

By (H_1'), we can follow the process of [35, Lemma 3.2] and obtain that

$$\rho_0 = 0 \text{ or } \rho_0 \geq S^{\frac{N}{2}}(\mu_1), \;\; \rho_\infty = 0 \text{ or } \rho_\infty \geq S^{\frac{N}{2}}(\mu_1). \tag{3.57}$$

Similarly we can obtain that

$$\overline{\rho}_0 = 0 \text{ or } \overline{\rho}_0 \geq S^{\frac{N}{2}}(\mu_2), \;\; \overline{\rho}_\infty = 0 \text{ or } \overline{\rho}_\infty \geq S^{\frac{N}{2}}(\mu_2). \tag{3.58}$$

For $x_j \in \mathbb{R}^N\backslash\{0\}$, if $h(x_j) \leq 0$, then we can argue as above and obtain that

$$\rho_j = 0 \text{ or } \rho_j \geq S^{\frac{N}{2}}; \overline{\rho}_j = 0 \text{ or } \overline{\rho}_j \geq S^{\frac{N}{2}}. \tag{3.59}$$

Next we consider $x_j \in \mathbb{R}^N\backslash\{0\}$ with $h(x_j) > 0$. Then one of the following holds:

(1) $\rho_j = \overline{\rho}_j = 0$;

(2) $\rho_j = 0$ and $\overline{\rho}_j > 0$;

(3) $\rho_j > 0$ and $\overline{\rho}_j = 0$;

(4) $\rho_j > 0$ and $\overline{\rho}_j > 0$.

If (3) holds, then (3.50) satisfies. By (3.56) and $x_j \neq 0$, we can obtain $\zeta_j - \rho_j \leq 0$. Then the Sobolev's inequality implies that either $\rho_j = 0$ or $\rho_j \geq S^{\frac{N}{2}}$. Thus, if (3) holds, we have $\rho_j \geq S^{\frac{N}{2}}$. Similarly, if (2) holds, we have $\overline{\rho}_j \geq S^{\frac{N}{2}}$. If (4) holds, we can not apply (3.50) but we claim

$$S(\rho_j^{\frac{2}{2^*}} + \overline{\rho}_j^{\frac{2}{2^*}}) \geq S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2). \tag{3.60}$$

Without loss of generality, we can assume that $\max\{\rho_j, \overline{\rho}_j\} \leq 2^{\frac{2^*}{2}} S^{\frac{N}{2}}$. Then by the Sobolev's inequality and (3.56), we have

$$S\rho_j^{\frac{2}{2^*}} - \rho_j - \alpha\Theta\rho_j^{\frac{\alpha}{2^*}}(2^{\frac{2^*}{2}}S^{\frac{N}{2}})^{\frac{\beta}{2^*}} \leq 0, \tag{3.61}$$

which is equivalent to $S \leq \rho_j^{1-\frac{2}{2^*}} + \alpha\Theta 2^{\frac{\beta}{2}} S^{\frac{(N-2)\beta}{4}} \rho_j^{\frac{\alpha-2}{2^*}}$. Consider the function $f(t) = t^{1-\frac{2}{2^*}} + \alpha\Theta 2^{\frac{\beta}{2}} S^{\frac{(N-2)\beta}{4}} t^{\frac{\alpha-2}{2^*}}$, which is increasing in $(0, +\infty)$ because of $\alpha \geq 2$. If $\Theta \leq \dfrac{1-(1-\varepsilon_1)^{\frac{2}{N}}}{2^{\frac{\beta}{2}}\alpha(1-\varepsilon_1)^{\frac{\alpha-2}{2^*}}}$, then we can compute $f\big((1-\varepsilon_1)S^{\frac{N}{2}}\big)$ as following:

$$f\big((1-\varepsilon_1)S^{\frac{N}{2}}\big) = \big[(1-\varepsilon_1)^{\frac{2}{N}} + \alpha\Theta 2^{\frac{\beta}{2}}(1-\varepsilon_1)^{\frac{\alpha-2}{2^*}}\big]S \leq S, \tag{3.62}$$

which implies that $\rho_j \geq (1-\varepsilon_1)S^{\frac{N}{2}}$ and $S\rho_j^{\frac{2}{2^*}} \geq (1-\varepsilon_1)^{\frac{N-2}{N}} S^{\frac{N}{2}}$. Similarly, if $\Theta \leq \dfrac{1-(1-\varepsilon_1)^{\frac{2}{N}}}{2^{\frac{\alpha}{2}}\beta(1-\varepsilon_1)^{\frac{\beta-2}{2^*}}}$, we have $S\overline{\rho}_j^{\frac{2}{2^*}} \geq (1-\varepsilon_1)^{\frac{N-2}{N}} S^{\frac{N}{2}}$. Hence, by (3.62) we have

$$S(\rho_j^{\frac{2}{2^*}} + \overline{\rho}_j^{\frac{2}{2^*}}) \geq 2(1-\varepsilon_1)^{\frac{N-2}{N}} S^{\frac{N}{2}} \geq S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2).$$

Step 2: We prove that either $u_n \to u_0$ or $v_n \to v_0$ strongly in $L^{2^*}(\mathbb{R}^N)$. If not, then there exist some $j_0 \in \mathcal{J} \cup \{0\} \cup \{\infty\}$ and $j_1 \in \mathcal{J} \cup \{0\} \cup \{\infty\}$ such that $\rho_{j_0} > 0, \overline{\rho}_{j_1} > 0$. If $j_0 \neq j_1$, then we have $\rho_{j_0} \geq S^{\frac{N}{2}}(\mu_1), \overline{\rho}_{j_1} \geq S^{\frac{N}{2}}(\mu_2)$ and obtain that

$$\begin{aligned} c &= \Phi(u_n, v_n) + o(1) \\ &= \frac{1}{N}\|(u_n, v_n)\|_{\mathbb{D}}^2 + o(1) \\ &\geq \frac{1}{N}S(\mu_1)\rho_{j_0}^{\frac{2}{2^*}} + \frac{1}{N}S(\mu_2)\overline{\rho}_{j_1}^{\frac{2}{2^*}} \\ &\geq \frac{1}{N}\big(S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2)\big), \end{aligned}$$

which contradicts with (3.10). If $j_0 = j_1 = j$ and $j \in \{0, \infty\}$, then we have $\rho_j \geq S^{\frac{N}{2}}(\mu_1), \overline{\rho}_j \geq S^{\frac{N}{2}}(\mu_2)$ and $c \geq \frac{1}{N}\big(S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2)\big)$, also a contradiction with (3.10). If $j_0 = j_1 = j$ and $j \notin \{0, \infty\}$, recall (3.54) and (3.60) we see that

$$c \geq \frac{1}{N}S\big(\rho_j^{\frac{2}{2^*}} + \overline{\rho}_j^{\frac{2}{2^*}}\big) \geq \frac{1}{N}\big(S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2)\big),$$

also a contradiction with (3.10). Thus, we have either $u_n \to u_0$ or $v_n \to v_0$ strongly in $L^{2^*}(\mathbb{R}^N)$.

Step 3: By the above arguments, under the conditions of $c < \frac{1}{N}\big(S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2)\big)$ and (3.54), we obtain that either ρ_j or $\overline{\rho}_j$ equals 0 for any $j \notin \{0, \infty\}$. Then (3.50) is satisfied. Further,

$$\rho_0 \geq S^{\frac{N}{2}}(\mu_1), \rho_\infty \geq S^{\frac{N}{2}}(\mu_1), \overline{\rho}_0 \geq S^{\frac{N}{2}}(\mu_2), \overline{\rho}_\infty \geq S^{\frac{N}{2}}(\mu_2) \tag{3.63}$$

and $\mathcal{J}$ is finite. For $j \in \mathcal{J}$ we have either

$$(\rho_j, \overline{\rho}_j) = (0,0) \ \text{ or } \big(\rho_j = 0, \overline{\rho}_j \geq S^{\frac{N}{2}}\big) \text{ or } \big(\rho_j \geq S^{\frac{N}{2}}, \overline{\rho}_j = 0\big). \tag{3.64}$$

Thus, the steps $3 \sim 5$ in Lemma 3.2 are valid here, and we finish the proof. $\square$

Consider $N = 4, \alpha = \beta = 2, \big(\dfrac{1-\mu_1}{1-\mu_2}\big)^{\frac{3}{2}} \geq \dfrac{1}{2}$. Similar to Lemma 3.3, we have the following lemma.

Lemma 3.4. *Consider $N = 4, \alpha = \beta = 2$. Assume $(H_1'), 0 \leq \mu_2 \leq \mu_1 < 1, \mu_1 + \mu_2 \neq 0, \big(\frac{1-\mu_1}{1-\mu_2}\big)^{\frac{3}{2}} \geq \frac{1}{2}$ and*

$$\begin{aligned}
\Theta &\leq \min\Big\{\frac{1-(1-\varepsilon_1)^{\frac{2}{N}}}{2^{\frac{\beta}{2}}\alpha(1-\varepsilon_1)^{\frac{\alpha-2}{2^*}}}, \frac{1-(1-\varepsilon_1)^{\frac{2}{N}}}{2^{\frac{\alpha}{2}}\beta(1-\varepsilon_1)^{\frac{\beta-2}{2^*}}}, \frac{2-\sqrt{2}}{4}\Big\} \\
&= \min\{\frac{1-(1-\varepsilon_1)^{\frac{1}{2}}}{4}, \frac{2-\sqrt{2}}{4}\} \text{ with } \varepsilon_1 \text{ satisfying (3.53)} \\
&= \min\{\frac{2-(1-\mu_1)^{\frac{3}{2}}-(1-\mu_2)^{\frac{3}{2}}}{8}, \frac{2-\sqrt{2}}{4}\}.
\end{aligned} \tag{3.65}$$

Let $\{(u_n, v_n)\} \subset \overline{\mathcal{N}}$ be a Palais-Smale sequence for $\overline{\Phi}|_{\overline{\mathcal{N}}}$ at level $c \in \mathbb{R}$. Then, there exists a constant C, such that $||(u_n, v_n)||_{\mathbb{D}} \leq C$ for all $n \in \mathbb{N}$ and that $\overline{\Phi}'(u_n, v_n) \to 0$ in the dual space $\mathbb{D}^*$. Moreover, if c satisfies

$$\frac{1}{N}S^{\frac{N}{2}}(\mu_2) < c < \frac{1}{N}S^{\frac{N}{2}}(\mu_2) + \inf_{(u,v)\in\mathcal{N}}\Phi(u,v) \leq \frac{1}{N}\big(S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2)\big); \tag{3.66}$$

$$c \neq \frac{l}{N}S^{\frac{N}{2}}(\mu_1) \ \text{ and } \ c \neq \frac{l}{N}S^{\frac{N}{2}} \text{ for all } l \in \mathbb{N}\backslash\{0\}, \tag{3.67}$$

then, up to a subsequence, $(u_n, v_n) \to (u_0, v_0)$ in $\mathbb{D}$.

Proof. Follow the processes of Lemma 3.2 and Lemma 3.3 carefully. First, by the similar argument as that in Lemma 3.3 we obtain (3.63) and (3.64). Then the remaining work is similar to the proof of steps $3 \sim 5$ in Lemma 3.2. We can see that the only difference is that from (3.41) to the end of Step 3. Thus, we can start as the following. Since $N = 4, \alpha = \beta = 2$, we have $S(\mu) = (1-\mu)^{\frac{3}{4}}S$, where S is the best constant for the Sobolev inequality in $\mathbb{R}^4$. Then (3.40) and (3.41) are equivalent to

$$\big(1-\mu_2\big)^{\frac{3}{4}}S \leq \|v_0\|^2_{L^4(\mathbb{R}^4)} + 2\Theta\big(1-\mu_2\big)^{\frac{3}{4}}S \tag{3.68}$$

and

$$\big(1-\mu_1\big)^{\frac{3}{4}}S \leq \|u_0\|^2_{L^4(\mathbb{R}^4)} + 2\Theta\big(1-\mu_2\big)^{\frac{3}{4}}S. \tag{3.69}$$

Consider $f(t) = t^{\frac{1}{2}} + 2\Theta(1-\mu_2)^{\frac{3}{4}}S$, which is increasing in $(0, +\infty)$. Assume $\Theta \leq \frac{2-\sqrt{2}}{4}, \mu_2 \leq \mu_1$, we have $f\big(\frac{1}{2}S^{\frac{N}{2}}(\mu_2)\big) \leq (1-\mu_1)^{\frac{3}{4}}S$. Hence, by (3.69), we have

$$\|u_0\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \geq \frac{1}{2}\big(1-\mu_2\big)^{\frac{3}{2}}S^2 = \frac{1}{2}S^{\frac{N}{2}}(\mu_2).$$

Similarly, by (3.68) we have $\|v_0\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \geq \frac{1}{2} S^{\frac{N}{2}}(\mu_2)$. Thus

$$\|u_0\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} + \|v_0\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \geq \frac{1}{2} S^{\frac{N}{2}}(\mu_2) + \frac{1}{2} S^{\frac{N}{2}}(\mu_2) = S^{\frac{N}{2}}(\mu_2),$$

a contradiction to (3.38). □

4 Nonexistence of the Nontrivial Least Energy Solution

We introduce the following notation:

$$C'_{\alpha,\beta} := \tfrac{1}{\beta}\big(\tfrac{S(\mu_2)}{S(\mu_1)} - 1\big) S^{\frac{4-(N-2)(\alpha+\beta-2)}{4}}(\mu_1) \ \text{ if } \ \mu_2 < \mu_1, \beta \geq 2;$$

$$C'_{\alpha,\beta} := \min\big\{\tfrac{2^{\frac{\beta-2}{2^*}}}{\beta}, \tfrac{2^{\frac{\alpha-2}{2^*}}}{\alpha}\big\}[1 - (\tfrac{1}{2})^{\frac{2}{N}}] S^{\frac{4-(N-2)(\alpha+\beta-2)}{4}}(\mu) \ \text{ if } \ \mu_2 = \mu_1 = \mu$$
$$\text{and } \alpha \geq 2, \beta \geq 2.$$

In particular, if $N = 3, 4$, we define the following simpler constants

$$C'_{\alpha,\beta} := \min\{0.09, 0.09 S^{-\frac{1}{2}}(\mu_1)\} \text{ if } N = 3, \mu_2 \leq \mu_1, \alpha \geq 2, \beta \geq 2.$$

$$C'_{\alpha,\beta} := \tfrac{2-\sqrt{2}}{2} \text{ if } N = 4, \alpha = \beta = 2, \mu_2 \leq \mu_1.$$

Theorem 4.1. *Assume that $\beta \geq 2, \mu_2 < \mu_1$ or $\alpha \geq 2, \beta \geq 2, \mu_2 = \mu_1 = \mu$. Suppose that*

$$\begin{cases} \text{either } h(x) \text{ satisfies } (H_1) \text{ if } \alpha + \beta < 2^*; \\ \text{or } h(x) \text{ satisfies } (H_1') \text{and } \mu_1 + \mu_2 \neq 0 \text{ if } \alpha + \beta = 2^*. \end{cases}$$

If further, $\Theta \leq \Theta_0 := C'_{\alpha,\beta}$ for respective cases of μ_1, μ_2 and α, β, then the least energy c of the system

$$c := \inf_{(u,v)\in\mathcal{N}} \Phi(u, v) = \frac{1}{N} S^{\frac{N}{2}}(\mu_1).$$

Moreover, c is achieved by and only by $(\pm z^{\mu_1}_\sigma, 0), \sigma > 0$ if $\mu_2 < \mu_1$ and by $(\pm z^{\mu}_\sigma, 0)$ and $(0, \pm z^{\mu}_\sigma)$ if $\mu_2 = \mu_1 = \mu \neq 0$ $\big($resp. $(\pm z_{\sigma,x_i}, 0)$ and $(0, \pm z_{\sigma,x_i})$ if $\mu_2 = \mu_1 = 0\big)$. That is, problem (1.1) has no nontrivial least energy solution.

Proof. First, we consider the case of $\mu_2 < \mu_1$. Since $(0, z^{\mu_2}_\sigma) \in \mathcal{N}$ and $\Phi(0, z^{\mu_2}_\sigma) = \frac{1}{N} S^{\frac{N}{2}}(\mu_2)$, we have that

$$c := \inf_{(u,v)\in\mathcal{N}} \Phi(u, v) \leq \frac{1}{N} S^{\frac{N}{2}}(\mu_2).$$

On the other hand, note that $(z_\sigma^{\mu_1}, 0) \in \mathcal{N}$ and $\Phi(z_\sigma^{\mu_1}, 0) = \frac{1}{N} S^{\frac{N}{2}}(\mu_1)$, we get that

$$c := \inf_{(u,v)\in\mathcal{N}} \Phi(u,v) \le \frac{1}{N} S^{\frac{N}{2}}(\mu_1). \tag{4.1}$$

Since $\mu_2 < \mu_1$, it follows that $S(\mu_1) < S(\mu_2)$ and that $c \le \frac{1}{N} S^{\frac{N}{2}}(\mu_1)$. If $c < \frac{1}{N} S^{\frac{N}{2}}(\mu_1)$, then by Lemma 2.1 we have $c > 0$. By Lemma 3.1, c can be obtained by some $(0,0) \not\equiv (\phi,\chi) \in \mathcal{N}$. Notice that the functional Φ is even, we have $(|\phi|,|\chi|) \in \mathcal{N}$ and $\Phi(\phi,\chi) = \Phi(|\phi|,|\chi|)$, hence $(|\phi|,|\chi|)$ is also a ground state for Φ. Without loss of generality, we can assume $\phi \ge 0, \chi \ge 0$. Moreover, $\phi \not\equiv 0$ and $\chi \not\equiv 0$. If not, $\phi \equiv 0$ implies that $\chi \not\equiv 0$ is a solution of

$$\begin{cases} -\Delta v - \mu_2 \frac{v}{|x|^2} = |v|^{2^*-2} v \quad \text{in } \mathbb{R}^N, \\ 0 \not\equiv v \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

then $\Phi(\phi,\chi) = \Phi(0,\chi) \ge \frac{1}{N} S^{\frac{N}{2}}(\mu_2) > \frac{1}{N} S^{\frac{N}{2}}(\mu_1) > c$, a contradiction. If $\chi \equiv 0, \phi \not\equiv 0$, then similarly we can get a contradiction. Recalling that

$$c = \Phi(\phi,\chi) = \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\|(\phi,\chi)\|_{\mathbb{D}}^2 + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)\Big(\|\phi\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} + \|\chi\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}\Big),$$

$$\int_{\mathbb{R}^N} \big(|\nabla\phi|^2 - \mu_1 \frac{\phi^2}{|x|^2}\big)dx = \int_{\mathbb{R}^N} |\phi|^{2^*} dx + \alpha \int_{\mathbb{R}^N} h(x)\phi^\alpha \chi^\beta dx,$$

$$\int_{\mathbb{R}^N} \big(|\nabla\chi|^2 - \mu_2 \frac{\chi^2}{|x|^2}\big)dx = \int_{\mathbb{R}^N} |\chi|^{2^*} dx + \beta \int_{\mathbb{R}^N} h(x)\phi^\alpha \chi^\beta dx.$$

Next, we discuss two cases according to the sign of $\int_{\mathbb{R}^N} h(x)\phi^\alpha\chi^\beta dx$.

- Case 1: $\displaystyle\int_{\mathbb{R}^N} h(x)\phi^\alpha\chi^\beta dx \le 0.$

It follows that

$$\int_{\mathbb{R}^N} \big(|\nabla\chi|^2 - \mu_2 \frac{\chi^2}{|x|^2}\big)dx \le \int_{\mathbb{R}^N} |\chi|^{2^*} dx,$$

which implies that $\|\chi\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \ge S^{\frac{N}{2}}(\mu_2)$. By the Hardy's inequality and the Sobolev's inequality, we have that

$$\begin{aligned} \Phi(\phi,\chi) &= \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\|(\phi,\chi)\|_{\mathbb{D}}^2 + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)\Big(\|\phi\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} + \|\chi\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}\Big) \\ &\ge \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\|\chi\|_{\mu_2}^2 + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)\|\chi\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \\ &\ge \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)S(\mu_2)\|\chi\|_{L^{2^*}(\mathbb{R}^N)}^{2} + \Big(\frac{1}{\alpha+\beta} - \frac{1}{2^*}\Big)\|\chi\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \\ &\ge \frac{1}{N} S^{\frac{N}{2}}(\mu_2) > \frac{1}{N} S^{\frac{N}{2}}(\mu_1) > c, \end{aligned}$$

a contradiction.

- Case 2: $\int_{\mathbb{R}^N} h(x)\phi^\alpha\chi^\beta dx > 0$.

Notice that

$$
\begin{aligned}
c=\Phi(\phi,\chi) &= \frac{1}{N}\big(\|\phi\|^{2^*}_{L^{2^*}(\mathbb{R}^N)}+\|\chi\|^{2^*}_{L^{2^*}(\mathbb{R}^N)}\big) \\
&\quad +(\alpha+\beta)(\frac{1}{2}-\frac{1}{\alpha+\beta})\int_{\mathbb{R}^N} h(x)\phi^\alpha\chi^\beta dx \\
&< \frac{1}{N}S^{\frac{N}{2}}(\mu_1),
\end{aligned}
\tag{4.2}
$$

we have

$$
\|\chi\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} < S^{\frac{N}{2}}(\mu_1), \|\phi\|^{2}_{L^{2^*}(\mathbb{R}^N)} < S^{\frac{N}{2}}(\mu_1). \tag{4.3}
$$

By the Hardy's inequality (or the Sobolev's inequality) and the Hölder's inequality, we see that

$$
\begin{aligned}
S(\mu_2)\|\chi\|^2_{L^{2^*}(\mathbb{R}^N)} &\le \|\chi\|^2_{\mu_2} \\
&= \|\chi\|^{2^*}_{L^{2^*}(\mathbb{R}^N)}+\beta\int_{\mathbb{R}^N} h(x)\phi^\alpha\chi^\beta dx \\
&\le \|\chi\|^{2^*}_{L^{2^*}(\mathbb{R}^N)}+\beta\int_{\mathbb{R}^N} h_+(x)\phi^\alpha\chi^\beta dx \\
&\le \|\chi\|^{2^*}_{L^{2^*}(\mathbb{R}^N)}+\beta\Theta S^{\frac{N-2}{4}\alpha}(\mu_1)\|\chi\|^{\beta}_{L^{2^*}(\mathbb{R}^N)}.
\end{aligned}
\tag{4.4}
$$

Denote $\sigma=\|\chi\|^{2^*}_{L^{2^*}(\mathbb{R}^N)}$, then

$$
S(\mu_2)\le \sigma^{1-\frac{2}{2^*}}+\beta\Theta S^{\frac{N-2}{4}\alpha}(\mu_1)\sigma^{\frac{\beta-2}{2^*}}. \tag{4.5}
$$

Let $f(\sigma)=\sigma^{1-\frac{2}{2^*}}+\beta\Theta S^{\frac{N-2}{4}\alpha}(\mu_1)\sigma^{\frac{\beta-2}{2^*}}$. Since $\beta\ge 2$, we have that $f(\sigma)$ is increasing in $(0,+\infty)$. If $\mu_2<\mu_1,\beta\ge 2,\Theta\le C'_{\alpha,\beta}$, the direct calculation implies that $f\big(S^{\frac{N}{2}}(\mu_1)\big)\le S(\mu_2)$. Then (4.5) implies that $\|\chi\|^{2^*}_{L^{2^*}(\mathbb{R}^N)}\ge S^{\frac{N}{2}}(\mu_1)$, a contradiction to (4.3). By the above arguments, we obtain that $c=\dfrac{1}{N}S^{\frac{N}{2}}(\mu_1)$ and obviously it is achieved by $(\pm z^{\mu_1}_\sigma,0),\sigma>0$.

Let (ϕ,χ) be a minimizer of $c=\frac{1}{N}S^{\frac{N}{2}}(\mu_1)$. If $\phi\not\equiv 0$ and $\chi\not\equiv 0$, it will lead to a contradiction if repeating the above arguments. Here, we may assume $\phi\equiv 0$, then $\Phi(\phi,\chi)\ge\frac{1}{N}S^{\frac{N}{2}}(\mu_2)>\frac{1}{N}S^{\frac{N}{2}}(\mu_1)=c$ if $\mu_2<\mu_1$. Hence, c is only achieved by $(\phi,0)$, where ϕ is a weak solution of

$$
\begin{cases}
-\Delta u-\mu_1\frac{u}{|x|^2}=|u|^{2^*-2}u \quad \text{in } \mathbb{R}^N,\\
0\not\equiv u\in D^{1,2}(\mathbb{R}^N).
\end{cases}
$$

By [31], it is easy to see that c is only achieved by $(\pm z^{\mu}_\sigma,0),\sigma>0$.

Second, we consider the case of $\mu_2 = \mu_1 = \mu, \alpha \geq 2, \beta \geq 2, \Theta \leq C'_{\alpha,\beta}$. The difference is the computation of $f\big(\frac{1}{2}S^{\frac{N}{2}}(\mu_1)\big)$:

$$\begin{aligned}
f\big(\frac{1}{2}S^{\frac{N}{2}}(\mu_1)\big) &= \big[\frac{1}{2}S^{\frac{N}{2}}(\mu_1)\big]^{\frac{2}{N}} + \beta\Theta S^{\frac{N-2}{4}\alpha}(\mu_1)\big[\frac{1}{2}S^{\frac{N}{2}}(\mu_1)\big]^{\frac{\beta-2}{2^*}} \\
&= (\frac{1}{2})^{\frac{2}{N}}S(\mu_1) + \beta\Theta(\frac{1}{2})^{\frac{\beta-2}{2^*}}S^{\frac{(N-2)(\alpha+\beta-2)}{4}}(\mu_1) \\
&\leq S(\mu_2) = S(\mu_1).
\end{aligned}$$

Then (4.5) implies that $\|\chi\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \geq \dfrac{1}{2}S^{\frac{N}{2}}(\mu)$. Similarly, we can obtain that

$$\|\phi\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \geq \frac{1}{2}S^{\frac{N}{2}}(\mu).$$

Then

$$\|\chi\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} + \|\phi\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \geq S^{\frac{N}{2}}(\mu) = S^{\frac{N}{2}}(\mu_1),$$

a contradiction to (4.2). Finally, it is easy to show that c is achieved by and only by semi-trivial solutions $(\pm z^{\mu}_{\sigma}, 0)$ and $(0, \pm z^{\mu}_{\sigma})$ if $\mu \neq 0$ (resp.$(\pm z_{\sigma,x_i}, 0)$ and $(0, \pm z_{\sigma,x_i})$ if $\mu = 0$). □

Proof of Theorem 1.1. It is a straightforward consequence of Theorem 4.1. □

5 The Existence of Mountain Pass Solutions

If $\min\{\alpha, \beta\} = 2$, we denote

$$C_{\alpha,\beta,\mu_1,\mu_2} := \min\Big\{\frac{S(\mu_1)}{2S^{\frac{N-2}{4}\beta}(\mu_2)}, \frac{S(\mu_2)}{2S^{\frac{N-2}{4}\alpha}(\mu_1)}\Big\} \tag{5.1}$$

Remark 5.1. *We observe that* $C_{\alpha,\beta,\mu_1,\mu_2} > \min\{0.3, 0.3S^{\frac{1}{2}}(\mu_1)\}$ *if* $N = 3$; $C_{\alpha,\beta,\mu_1,\mu_2} > \frac{\sqrt{2}}{4}$ *if* $N = 4$ *and* $\alpha = \beta = 2$.

Remark 5.2. *For* $\alpha > 2$, *we can see that the indefinite sign of* $h(x)$ *has no effect on the proof of Theorem 2.2 in [1]. Thus, for all* $\sigma > 0, z^{\mu_2}_{\sigma}$ *is a local minimum point of* Φ *in* $\mathcal{N}$. *Denote*

$$\mathcal{N}_{\mu_2} = \big\{u \in D^{1,2}(\mathbb{R}^N)\backslash\{0\} : \|u\|^2_{\mu_2} = \|u\|^{2^*}_{L^{2^*}(\mathbb{R}^N)}\big\},$$

which is the corresponding Nehari manifold of I_{μ_2}, *where* I_μ *is defined in (3.2). For* $\alpha = 2, \mu_2 > 0$, *we can obtain that* $\big($*for the details we refer to [1, Theorem 2.2] or [35, Lemma 3.4]*$\big)$ $(\phi, z^{\mu_2}_{\sigma} + \chi) \in \mathcal{N}$, *where* z^{μ}_{σ} *is defined in (1.5). Let*

$t>0$ satisfy $t(z_\sigma^{\mu_2}+\chi)\in\mathcal{N}_{\mu_2}$, then

$$\begin{aligned}
&\Phi(\phi, z_\sigma^{\mu_2}+\chi)-\Phi(0,t(z_\sigma^{\mu_2}+\chi))\\
&=\big(\frac{1}{2}\|\phi\|^2_{\mu_1}-\int_{\mathbb{R}^N}h(x)|\phi|^2|z_\sigma^{\mu_2}+\chi|^\beta\big)(1+o(1))\\
&\geq\Big(\frac{1}{2}S(\mu_1)\|\phi\|^2_{L^{2^*}(\mathbb{R}^N)}-\Theta\|\phi\|^2_{L^{2^*}(\mathbb{R}^N)}\|z_\sigma^{\mu_2}+\chi\|^\beta_{L^{2^*}(\mathbb{R}^N)}\Big)(1+o(1))\\
&=\Big(\frac{1}{2}S(\mu_1)-\Theta\|z_\sigma^{\mu_2}+\chi\|^\beta_{L^{2^*}(\mathbb{R}^N)}\Big)\|\phi\|^2_{L^{2^*}(\mathbb{R}^N)}(1+o(1))\\
&\to\Big(\frac{1}{2}S(\mu_1)-\Theta S^{\frac{N-2}{4}\beta}(\mu_2)\Big)\|\phi\|^2_{L^{2^*}(\mathbb{R}^N)}(1+o(1))
\end{aligned}$$

as $\|(\phi,\chi)\|_{\mathbb{D}}\to 0$. Thus, when $\Theta<C_{\alpha,\beta,\mu_1,\mu_2}$ (see (5.1)), we have that

$$\frac{1}{2}S(\mu_1)-\Theta S^{\frac{N-2}{4}\beta}(\mu_2)>0.$$

It follows that

$$\Phi(\phi, z_\sigma^{\mu_2}+\chi)-\Phi(0,t(z_\sigma^{\mu_2}+\chi))>0 \tag{5.2}$$

provided that $(\phi, z_\sigma^{\mu_2}+\chi)\in\mathcal{N}$ is sufficiently closed to $(0,z_\sigma^{\mu_2})$. On the other hand, since $z_\sigma^{\mu_2}$ is a minimizer of I_{μ_2} on $\mathcal{N}_{\mu_2}$, we have

$$\Phi(0,t(z_\sigma^{\mu_2}+\chi)-\Phi(0,z_\sigma^{\mu_2}))=I_{\mu_2}\big(t(z_\sigma^{\mu_2}+\chi)\big)-I_{\mu_2}(z_\sigma^{\mu_2})\geq 0. \tag{5.3}$$

From (5.2) and (5.3), we conclude that $\Phi(\phi, z_\sigma^{\mu_2}+\chi)-\Phi(0,z_\sigma^{\mu_2})\geq 0$ for any $(\phi, z_\sigma^{\mu_2}+\chi)\in\mathcal{N}$ sufficiently close to $(0,z_\sigma^{\mu_2})$(with strictly inequality hold outside the manifold $\{0\}\times Z_{\mu_2}$, where $Z_{\mu_2}=\{z_\sigma^{\mu_2},\sigma>0\}$), i.e., $(0,z_\sigma^{\mu_2})$ is a local minimum point of Φ in $\mathcal{N}$. Similarly, for $\alpha=2,\mu_2=0$, we replace $z_\sigma^{\mu_2}$ by z_{σ,x_i}, where $z_{\sigma,x_i}=\sigma^{-\frac{N-2}{2}}z_1(\frac{x-x_i}{\sigma}),\sigma>0,x_i\in\mathbb{R}^N$, and $z_\sigma(x)=z_{\sigma,0}$, $z_{1,0}(x)=\dfrac{[N(N-2)]^{\frac{N-2}{4}}}{[1+|x|^2]^{\frac{N-2}{2}}}$, we can obtain that $\Phi(\phi,z_{\sigma,x_i}+\chi)-\Phi(0,z_{\sigma,x_i})\geq 0$ for any $(\phi,z_{\sigma,x_i}+\chi)\in\mathcal{N}$ sufficiently closed to $(0,Z_{\sigma,x_i})$ (with strictly inequality holding outside the manifold $\{0\}\times Z_0$, where $Z_0=\{z_{\sigma,x_i},\sigma>0,x_i\in\mathbb{R}^N\}$), i.e., $(0,z_{\sigma,x_i})$ is a local minimum point of Φ in $\mathcal{N}$. Similarly, if $\beta>2$ or $\beta=2$ with $\Theta<C_{\alpha,\beta,\mu_1,\mu_2}$ (see (5.1)), we can obtain that $(z_\sigma^{\mu_1},0)$ is a local minimum. So does $(z_{\sigma,x_i},0)$ if $\mu_1=0$ and either $\beta>2$ or $\beta=2$ with $\Theta<C_{\alpha,\beta,\mu_1,\mu_2}$. □

By Remark 5.2, the semi-trivial solutions $(0,z_\sigma^{\mu_2})$ or $(0,z_{\sigma,x_i})$ turns out to be local minimum points for the functional $\Phi\big|_{\mathcal{N}}$, which consequently exhibits a mountain pass geometry.

Next, our goal is to construct a mountain pass level for the functional on the Nehari manifold at which the Palais-Smale condition holds in view of Lemmas 3.2, 3.3 and 3.4. For the simplicity, when $\mu=0$, we also use the notation z_1^0 instead of $z_{1,0}$. When $\alpha>2$ or $\alpha=2$ with $\Theta<C_{\alpha,\beta,\mu_1,\mu_2}$ (see (5.1)), by

Remark 5.2, we know that $(z_1^{\mu_1}, 0)$ is a local minimum. Similarly, when $\beta > 2$ or $\beta = 2$ with $\Theta < C_{\alpha,\beta,\mu_1,\mu_2}$, $(0, z_1^{\mu_2})$ is a local minimum. By Theorem 4.1, when $\mu_2 < \mu_1, \Theta \leq C'_{\alpha,\beta}$ or $\mu_2 = \mu_1, \Theta \leq C'_{\alpha,\beta}$, we have $\inf\limits_{(u,v)\in\mathcal{N}} \Phi(u,v) = \frac{1}{N} S^{\frac{N}{2}}(\mu_1)$. Since Φ is even, for $(u,v) \in \mathcal{N}$, we have $(|u|,|v|) \in \mathcal{N}$. Thus, $(|u|,|v|) \in \mathcal{N} \cap \overline{\mathcal{N}}$ and $\overline{\Phi}(|u|,|v|) = \Phi(|u|,|v|) = \Phi(u,v)$, then $\inf\limits_{(u,v)\in\overline{\mathcal{N}}} \overline{\Phi}(u,v) \leq \dfrac{1}{N} S^{\frac{N}{2}}(\mu_1)$. Assume $\inf\limits_{(u,v)\in\overline{\mathcal{N}}} \overline{\Phi}(u,v) < \dfrac{1}{N} S^{\frac{N}{2}}(\mu_1) \leq \dfrac{1}{N} S^{\frac{N}{2}}(\mu_2)$, similar to Lemma 3.1, we can prove that the infimum can be achieved by some (u_0, v_0), and the minimizer is a critical point of $\overline{\Phi}$. It is easy to see $u_0 \geq 0, v_0 \geq 0$, thus $(u_0, v_0) \in \mathcal{N} \cap \overline{\mathcal{N}}$ and $\overline{\Phi}(u_0, v_0) = \Phi(u_0, v_0) \geq \frac{1}{N} S^{\frac{N}{2}}(\mu_1)$, a contradiction. Thus, we have

$$\inf_{(u,v)\in\overline{\mathcal{N}}} \overline{\Phi}(u,v) = \frac{1}{N} S^{\frac{N}{2}}(\mu_1) \tag{5.4}$$

when $\mu_2 \leq \mu_1, \Theta \leq C'_{\alpha,\beta}$.

Next, we consider the set of the paths in $\overline{\mathcal{N}}$ joining $(z_\sigma^{\mu_1}, 0)$ with $(0, z_\sigma^{\mu_2})$, i.e.,

$$\Sigma = \left\{ \begin{array}{c} \gamma = (\gamma_1, \gamma_2) : [0,1] \to \overline{\mathcal{N}} \text{ continous such that} \\ \\ \gamma_1(0) = 0, \gamma_1(1) = z_\sigma^{\mu_1}, \gamma_2(0) = z_\sigma^{\mu_2}, \gamma_2(1) = 0 \end{array} \right\},$$

and define the associated mountain pass level

$$C_{MP} := \inf_{\gamma\in\Sigma} \max_{t\in[0,1]} \overline{\Phi}(\gamma(t)). \tag{5.5}$$

5.1 The case of $N = 3, \frac{1}{2} < \frac{1-4\mu_1}{1-4\mu_2}, S^{\frac{N}{2}}(\mu_2) + S^{\frac{N}{2}}(\mu_1) \leq S^{\frac{N}{2}}$.

If $\mu_2 < \mu_1$, we define

$$M_1 := 1 - \frac{2\sqrt{2}(1-4\mu_1)}{[(1-4\mu_2)^{\frac{2}{3}} + (1-4\mu_1)^{\frac{2}{3}}]^{\frac{3}{2}}}, \tag{5.6}$$

$$M_2 := \frac{2(1-4\mu_1)}{1-4\mu_2} - 1, \tag{5.7}$$

$$M_3 := \frac{1}{2} \min\{M_1, M_2\}, \tag{5.8}$$

then we have

$$\frac{2}{N}(1-M_3)\Big(\frac{S(\mu_2)+S(\mu_1)}{2}\Big)^{\frac{N}{2}} > \frac{2}{N} S^{\frac{N}{2}}(\mu_1) > \frac{1+M_3}{N} S^{\frac{N}{2}}(\mu_2). \tag{5.9}$$

If $\mu_2 = \mu_1 = \mu$, we define

$$M_3 := \frac{1}{4}, \tag{5.10}$$

then we have

$$\frac{2}{N}(1-M_3)\big(\frac{S(\mu_1)+S(\mu_2)}{2}\big)^{\frac{N}{2}} > \frac{1+M_3}{N}S^{\frac{N}{2}}(\mu_2). \tag{5.11}$$

Define

$$M_4 := \frac{1-\frac{1}{2}(1-M_3)^{\frac{2}{N}}}{(\alpha+\beta)(\frac{1}{2})^{\frac{(N-2)(\alpha+\beta-2)}{4}}} > \frac{\sqrt{2}}{12}. \tag{5.12}$$

Lemma 5.1. *Assume $N=3, \alpha\geq 2, \beta\geq 2, \alpha+\beta\leq 2^*, \frac{1}{2}<\frac{1-4\mu_1}{1-4\mu_2}$ such that $S^{\frac{N}{2}}(\mu_1)+S^{\frac{N}{2}}(\mu_2)\leq S^{\frac{N}{2}}$. The weight function $h(x)$ satisfies (H_2) and $\begin{cases}(H_1) \text{ if } \alpha+\beta<2^*\\ (H_1') \text{ if } \alpha+\beta=2^*\end{cases}$. Moreover, if $\min\{\alpha,\beta\}=2$, we assume that $\Theta < C_{\alpha,\beta,\mu_1,\mu_2}$ (see (5.1)), where Θ is defined in (1.12). Then, if $\Theta\leq \min\{M_4, C'_{\alpha,\beta}\}$, the functional $\overline{\Phi}$ exhibits a mountain pass geometry and the mountain pass level satisfies (3.10) and (3.11).*

Proof. We claim that when $\widetilde{\alpha}\leq M_4$, we have

$$\max_{t\in[0,1]}\overline{\Phi}\big(\gamma(t)\big)\geq \frac{2}{N}(1-M_3)\Big(\frac{S(\mu_1)+S(\mu_2)}{2}\Big)^{\frac{N}{2}} \text{ for all } \gamma\in\Sigma. \tag{5.13}$$

Let $(\gamma_1,\gamma_2)\in\Sigma$, since $(\gamma_1(t),\gamma_2(t))\in\overline{\mathcal{N}}$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla\gamma_1(t)|^2dx-\mu_1\int_{\mathbb{R}^N}\frac{\gamma_1^2(t)}{|x|^2}dx+\int_{\mathbb{R}^N}|\nabla\gamma_2(t)|^2dx-\mu_2\int_{\mathbb{R}^N}\frac{\gamma_2^2(t)}{|x|^2}dx\\ &=\int_{\mathbb{R}^N}\big(\gamma_1(t)\big)_+^{2^*}dx+\int_{\mathbb{R}^N}\big(\gamma_2(t)\big)_+^{2^*}dx+(\alpha+\beta)\int_{\mathbb{R}^N}h(x)\big(\gamma_1(t)\big)_+^{\alpha}\big(\gamma_2(t)\big)_+^{\beta}dx\end{aligned}$$

and

$$\begin{aligned}\overline{\Phi}\big(\gamma_1(t),\gamma_2(t)\big) &= \frac{1}{N}\Big(\int_{\mathbb{R}^N}\big(\gamma_1(t)\big)_+^{2^*}dx+\int_{\mathbb{R}^N}\big(\gamma_2(t)\big)_+^{2^*}dx\Big)\\ &\quad+\frac{\alpha+\beta-2}{2}\int_{\mathbb{R}^N}h(x)\big(\gamma_1(t)\big)_+^{\alpha}\big(\gamma_2(t)\big)_+^{\beta}dx\end{aligned} \tag{5.14}$$

or

$$\begin{aligned}\overline{\Phi}\big(\gamma_1(t),\gamma_2(t)\big) &= \big(\frac{1}{2}-\frac{1}{\alpha+\beta}\big)\|(\gamma_1(t),\gamma_2(t))\|_{\mathbb{D}}^2+\big(\frac{1}{\alpha+\beta}-\frac{1}{2^*}\big)\\ &\quad\Big(\int_{\mathbb{R}^N}\big(\gamma_1(t)\big)_+^{2^*}dx+\int_{\mathbb{R}^N}\big(\gamma_2(t)\big)_+^{2^*}dx\Big)\end{aligned} \tag{5.15}$$

or

$$\begin{aligned}\overline{\Phi}\big(\gamma_1(t),\gamma_2(t)\big) &= \frac{1}{N}\|(\gamma_1(t),\gamma_2(t))\|_{\mathbb{D}}^2-\big(1-\frac{\alpha+\beta}{2^*}\big)\\ &\quad\int_{\mathbb{R}^N}h(x)\big(\gamma_1(t)\big)_+^{\alpha}\big(\gamma_2(t)\big)_+^{\beta}dx.\end{aligned} \tag{5.16}$$

Denote $f_i(t) = \int_{\mathbb{R}^N} \big(\gamma_i(t)\big)_+^{2^*} dx$ for $i = 1, 2$, then

$$f_1(0) = 0 < f_2(0) = \int_{\mathbb{R}^N} (z_\sigma^{\mu_2})^{2^*} dx \text{ and } f_1(1) = \int_{\mathbb{R}^N} (z_\sigma^{\mu_1})^{2^*} dx > f_2(1) = 0,$$

hence, by the continuity, there exists some $t_0 \in (0, 1)$ such that $f_1(t_0) = f_2(t_0) > 0$. From the definition of $S(\mu_1)$ and $S(\mu_2)$ and the Hölder's inequality, we have

$$\begin{aligned}
&S(\mu_1)\Big(\int_{\mathbb{R}^N} \big(\gamma_1(t_0)\big)_+^{2^*}\Big)^{\frac{2}{2^*}} + S(\mu_2)\Big(\int_{\mathbb{R}^N} \big(\gamma_2(t_0)\big)_+^{2^*}\Big)^{\frac{2}{2^*}} \\
&\le \int_{\mathbb{R}^N} |\nabla\gamma_1(t_0)|^2 dx - \mu_1 \int_{\mathbb{R}^N} \frac{\gamma_1^2(t_0)}{|x|^2} dx + \int_{\mathbb{R}^N} |\nabla\gamma_2(t_0)|^2 dx - \\
&\quad \mu_2 \int_{\mathbb{R}^N} \frac{\gamma_2^2(t_0)}{|x|^2} dx \\
&= \int_{\mathbb{R}^N} \big(\gamma_1(t_0)\big)_+^{2^*} dx + \int_{\mathbb{R}^N} \big(\gamma_2(t_0)\big)_+^{2^*} dx + \\
&\quad (\alpha+\beta) \int_{\mathbb{R}^N} h(x)\big(\gamma_1(t_0)\big)_+^{\alpha}\big(\gamma_2(t_0)\big)_+^{\beta} dx \\
&\le \int_{\mathbb{R}^N} \big(\gamma_1(t_0)\big)_+^{2^*} dx + \int_{\mathbb{R}^N} \big(\gamma_2(t_0)\big)_+^{2^*} dx + \\
&\quad (\alpha+\beta)\Theta\Big(\int_{\mathbb{R}^N} \big(\gamma_1(t_0)\big)_+^{2^*} dx\Big)^{\frac{\alpha}{2^*}}\Big(\int_{\mathbb{R}^N} \big(\gamma_2(t_0)\big)_+^{2^*} dx\Big)^{\frac{\beta}{2^*}}.
\end{aligned}$$

Set $\sigma = \int_{\mathbb{R}^N} \big(\gamma_1(t_0)\big)_+^{2^*} dx = \int_{\mathbb{R}^N} \big(\gamma_2(t_0)\big)_+^{2^*} dx$, then we obtain that

$$\big(S(\mu_1) + S(\mu_2)\big)\sigma^{\frac{2}{2^*}} \le \sigma + (\alpha+\beta)\Theta\sigma^{\frac{\alpha+\beta}{2^*}} \tag{5.17}$$

and it is equivalent to

$$S(\mu_1) + S(\mu_2) \le \sigma^{1-\frac{2}{2^*}} + (\alpha+\beta)\Theta\sigma^{\frac{\alpha+\beta-2}{2^*}}. \tag{5.18}$$

Denote $g(\sigma) := \sigma^{1-\frac{2}{2^*}} + (\alpha+\beta)\Theta\sigma^{\frac{\alpha+\beta-2}{2^*}}$, which is increasing in $(0, \infty)$. Recall that $\Theta \le M_4$, it follows that

$$\begin{aligned}
&g\Big((1 - M_3)\big(\frac{S(\mu_1) + S(\mu_2)}{2}\big)^{\frac{N}{2}}\Big) \\
&= (1 - M_3)^{\frac{2}{N}} \frac{S(\mu_1) + S(\mu_2)}{2} + (\alpha+\beta)\Theta\Big(\frac{S(\mu_1) + S(\mu_2)}{2}\Big)^{\frac{(\alpha+\beta-2)(N-2)}{4}} \\
&\le S(\mu_1) + S(\mu_2),
\end{aligned}$$

which implies that

$$\sigma \ge (1 - M_3)\big(\frac{S(\mu_1) + S(\mu_2)}{2}\big)^{\frac{N}{2}},$$

then

$$
\begin{aligned}
\overline{\Phi}(\gamma(t_0)) &\geq 2\Big[(\frac{1}{2}-\frac{1}{\alpha+\beta})(\frac{S(\mu_1)+S(\mu_2)}{2})\sigma^{\frac{2}{2^*}}+(\frac{1}{\alpha+\beta}-\frac{1}{2^*})\sigma\Big] \\
&\geq 2\Big[(\frac{1}{2}-\frac{1}{\alpha+\beta})(1-M_3)^{\frac{2}{2^*}}(\frac{S(\mu_1)+S(\mu_2)}{2})^{\frac{N}{2}} \\
&\quad +(\frac{1}{\alpha+\beta}-\frac{1}{2^*})(1-M_3)(\frac{S(\mu_1)+S(\mu_2)}{2})^{\frac{N}{2}}\Big] \\
&> \frac{2}{N}(1-M_3)(\frac{S(\mu_1)+S(\mu_2)}{2})^{\frac{N}{2}},
\end{aligned}
$$

we prove the claim (5.13) and obtain that

$$
C_{MP} > \frac{1+M_3}{N}S^{\frac{N}{2}}(\mu_2) = (1+M_3)\overline{\Phi}(0, z_\sigma^{\mu_2}),
$$

and hence $\overline{\Phi}$ exhibits a mountain pass geometry. In particular, $C_{MP} > \frac{2}{N}S^{\frac{N}{2}}(\mu_1)$ if $\mu_2 < \mu_1$; $C_{MP} > \frac{1+M_3}{N}S^{\frac{N}{2}}(\mu_1)$ if $\mu_2 = \mu_1$.

Next, we consider a special path $\gamma(t) = k(t)\big(\gamma_1(t), \gamma_2(t)\big) \in \Sigma, t \in [0,1]$ with $\gamma_1(t) = t^{\frac{1}{2}} z_\sigma^{\mu_1}$ and $\gamma_2(t) = (1-t)^{\frac{1}{2}} z_\sigma^{\mu_2}$, where $k(t)$ is a positive function such that $k(t)\big(\gamma_1(t), \gamma_2(t)\big) \in \mathcal{N} \cap \overline{\mathcal{N}}$. By the definition of the Nehari manifold, $k(t)$ is well defined and unique. For the simplicity, we set

$$
a := \|z_\sigma^{\mu_1}\|_{\mu_1}^2 = \int_{\mathbb{R}^N} |z_\sigma^{\mu_1}|^{2^*} dx = S^{\frac{N}{2}}(\mu_1)
$$

and

$$
b := \|z_\sigma^{\mu_2}\|_{\mu_2}^2 = \int_{\mathbb{R}^N} |z_\sigma^{\mu_2}|^{2^*} dx = S^{\frac{N}{2}}(\mu_2).
$$

Since $k(t)\big(\gamma_1(t), \gamma_2(t)\big) \in \mathcal{N} \cap \overline{\mathcal{N}}$, then by (H_2), we obtain

$$
\begin{aligned}
&\|(t^{\frac{1}{2}} z_\sigma^{\mu_1}, (1-t)^{\frac{1}{2}} z_\sigma^{\mu_2})\|_{\mathbb{D}}^2 \\
&= k^{2^*-2}(t)\big((1-t)^{\frac{2^*}{2}} a + t^{\frac{2^*}{2}} b\big) \\
&\quad +(\alpha+\beta)k^{\alpha+\beta-2}(t)(1-t)^{\frac{\alpha}{2}} t^{\frac{\beta}{2}} \int_{\mathbb{R}^N} h(x)|z_\sigma^{\mu_1}|^\alpha |z_\sigma^{\mu_2}|^\beta dx \\
&> k^{2^*-2}(t)\big((1-t)^{\frac{2^*}{2}} a + t^{\frac{2^*}{2}} b\big) \text{ for } 0 < t < 1.
\end{aligned}
$$

Hence,

$$
k(t) < \Big[\frac{(1-t)a + tb}{(1-t)^{\frac{2^*}{2}} a + t^{\frac{2^*}{2}} b}\Big]^{\frac{N-2}{4}} \text{ for all } 0 < t < 1 \tag{5.19}
$$

and $k(0) = k(1) = 1$. Combine (H_2) and (5.16), it follows that

$$
\overline{\Phi}\big(k(t)(\gamma_1(t), \gamma_2(t))\big) < \frac{k^2(t)}{N}\big((1-t)a + tb\big) \text{ for all } 0 < t < 1,
$$

hence,

$$C_{MP} = \inf_{\gamma \in \Sigma} \max_{t \in [0,1]} \overline{\Phi}\big(\gamma(t)\big) \leq \max_{t \in [0,1]} \overline{\Phi}(k(t)(\gamma_1(t), \gamma_2(t))). \tag{5.20}$$

By (5.16) and (5.19), we have

$$\overline{\Phi}\big(k(t)(\gamma_1(t), \gamma_2(t))\big) < \frac{1}{N}\Big[\frac{(1-t)a + tb}{(1-t)^{\frac{2^*}{2}}a + t^{\frac{2^*}{2}}b}\Big]^{\frac{N-2}{N}}\big((1-t)a+tb\big) \text{ for all } t \in (0,1).$$

After a direct computation, the right-hand side achieves its maximum at $t = \frac{1}{2}$ and the maximum is $\frac{1}{N}(a+b)$. Combining with (5.20), we obtain that

$$C_{MP} \leq \max_{t \in [0,1]} \overline{\Phi}(k(t)(\gamma_1(t), \gamma_2(t))) < \frac{1}{N}\big(S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2)\big). \tag{5.21}$$

Recall that $\frac{1}{2} < \frac{1-4\mu_1}{1-4\mu_2}$, we have $S^{\frac{N}{2}}(\mu_2) < 2S^{\frac{N}{2}}(\mu_1)$. Therefore, from the above arguments we obtain that

$$\begin{aligned}\frac{1+M_3}{2}S^{\frac{N}{2}}(\mu_2) < \frac{2}{N}S^{\frac{N}{2}}(\mu_1) < C_{MP} < \frac{1}{N}(S^{\frac{N}{2}}(\mu_1) + S^{\frac{N}{2}}(\mu_2))\\ < \min\{\frac{1}{N}S^{\frac{N}{2}}, \frac{3}{N}S^{\frac{N}{2}}(\mu_1)\} \text{ if } \mu_2 < \mu_1;\end{aligned}$$

and

$$\frac{1+M_3}{N}S^{\frac{N}{2}}(\mu) < C_{MP} < \frac{2}{N}S^{\frac{N}{2}}(\mu) \leq \frac{1}{N}S^{\frac{N}{2}} \text{ if } \mu_2 = \mu_1 = \mu.$$

Combining (5.4), it follows that both (3.10) and (3.11) are satisfied. □

5.2 The case of $N = 4, \frac{1}{2} < \big(\frac{1-\mu_1}{1-\mu_2}\big)^{\frac{3}{2}}, S^{\frac{N}{2}}(\mu_2) + S^{\frac{N}{2}}(\mu_1) \leq S^{\frac{N}{2}}$.

In this case, we only consider $\alpha = \beta = 2$. If $\mu_2 < \mu_1$, define

$$M_1' := 1 - \frac{4(1-\mu_1)^{\frac{3}{2}}}{[(1-\mu_2)^{\frac{3}{4}} + (1-\mu_1)^{\frac{3}{4}}]^2}, \tag{5.22}$$

$$M_2' := 2\big(\frac{1-\mu_1}{1-\mu_2}\big)^{\frac{3}{2}} - 1, \tag{5.23}$$

$$M_3' := \frac{1}{2}\min\{M_1', M_2'\}, \tag{5.24}$$

then we have

$$\frac{2}{N}(1-M_3')\big(\frac{S(\mu_2)+S(\mu_1)}{2}\big)^{\frac{N}{2}} > \frac{2}{N}S^{\frac{N}{2}}(\mu_1) > \frac{1+M_3'}{N}S^{\frac{N}{2}}(\mu_2). \tag{5.25}$$

If $\mu_2 = \mu_1 = \mu$, we define

$$M_3' := \frac{1}{4}, \tag{5.26}$$

then we have

$$\frac{2}{N}(1-M_3')\Big(\frac{S(\mu_1)+S(\mu_2)}{2}\Big)^{\frac{N}{2}} > \frac{1+M_3'}{N}S^{\frac{N}{2}}(\mu_2). \tag{5.27}$$

Define

$$M_4' := \frac{1-\frac{1}{2}(1-M_3')^{\frac{2}{N}}}{(\alpha+\beta)(\frac{1}{2})^{\frac{(N-2)(\alpha+\beta-2)}{4}}}. \tag{5.28}$$

If $\alpha=\beta=2$, then $M_4' = \frac{1}{2}-\frac{1}{4}(1-M_3')^{\frac{1}{2}} > \frac{1}{4}$. Similar to Lemma 5.1, we have the following

Lemma 5.2. *Assume $N=4, \alpha=\beta=2, \frac{1}{2} < \big(\frac{1-\mu_1}{1-\mu_2}\big)^{\frac{3}{2}}$ such that $S^{\frac{N}{2}}(\mu_1)+S^{\frac{N}{2}}(\mu_2) \le S^{\frac{N}{2}}$. Suppose that the weight function $h(x)$ satisfies (H_2)-(H_1') and that $\Theta < C_{\alpha,\beta,\mu_1,\mu_2}$ (see (5.1)). If moreover, $\Theta \le \min\{M_4', C'_{\alpha,\beta}\}$, then $\overline{\Phi}$ has a mountain pass geometry and the mountain pass level satisfies (3.66) and (3.67).*

Proof. Since we always assume $\alpha\ge 2, \beta\ge 2, \alpha+\beta\le 2^*$, when $N=4$, the only possibility is that $\alpha=\beta=2, \alpha+\beta=4=2^*$. Thus we required (H_1'). Note that $\frac{1}{2} < \big(\frac{1-\mu_1}{1-\mu_2}\big)^{\frac{3}{2}}$ implies $S^{\frac{N}{2}}(\mu_2) < 2S^{\frac{N}{2}}(\mu_1)$. We can follow carefully the processes of Lemma 5.1 and obtain the results. Here we omit the details. □

5.3 The case of $N=3, \frac{1}{2} < \frac{1-4\mu_1}{1-4\mu_2}, 2\big(\frac{S(\mu_2)+S(\mu_1)}{2}\big)^{\frac{N}{2}} > S^{\frac{N}{2}}$.

In this case, if $\mu_2<\mu_1$, M_3 is redefined as

$$M_3 := \frac{1}{2}\Big\{M_1, M_2, 1-\frac{\sqrt{2}}{[(1-4\mu_1)^{\frac{2}{3}}+(1-4\mu_2)^{\frac{2}{3}}]^{\frac{3}{2}}}\Big\}, \tag{5.29}$$

where M_1, M_2 are defined in (5.6) and (5.7). Then (5.9) satisfies and

$$2(1-M_3)\Big(\frac{S(\mu_1)+S(\mu_2)}{2}\Big)^{\frac{N}{2}} > S^{\frac{N}{2}}. \tag{5.30}$$

If $\mu_2=\mu_1=\mu$, M_3 is redefined as

$$M_3 := \frac{1}{2}-\frac{1}{4(1-4\mu)}, \tag{5.31}$$

then (5.11) and (5.30) are satisfied. Here we define

$$M_4 := \frac{1-\frac{1}{2}(1-M_3)^{\frac{2}{N}}}{(\alpha+\beta)(\frac{1}{2})^{\frac{(N-2)(\alpha+\beta-2)}{4}}} > \frac{\sqrt{2}}{12}. \tag{5.32}$$

Then we have the following result:

Lemma 5.3. *Assume $N=3, \alpha \geq 2, \beta \geq 2, \alpha+\beta \leq 2^*, \frac{1}{2} < \frac{1-4\mu_1}{1-4\mu_2}$ such that $2\big(\dfrac{S(\mu_2)+S(\mu_1)}{2}\big)^{\frac{N}{2}} > S^{\frac{N}{2}}$. The weight function $h(x)$ satisfies (H_2) and $\begin{cases}(H_1) \text{ if } \alpha+\beta<2^* \\ (H_1') \text{ if } \alpha+\beta=2^*\end{cases}$. In particular, when $\min\{\alpha,\beta\}=2$, we assume that $\Theta < C_{\alpha,\beta,\mu_1,\mu_2}$ (see (5.1)). Then, if $\Theta \leq \min\{M_4, C'_{\alpha,\beta}\}$, $\overline{\Phi}$ has a mountain pass geometry and the mountain pass level satisfies both (3.10) and (3.11).*

Proof. Analogous to Lemma 5.1, $\overline{\Phi}$ exhibits a mountain pass geometry at level C_{MP} satisfying

$$\frac{2}{N}(1-M_3)\big(\frac{S(\mu_1)+S(\mu_2)}{2}\big)^{\frac{N}{2}} \leq C_{MP} < \frac{1}{N}S^{\frac{N}{2}}(\mu_1)+\frac{1}{N}S^{\frac{N}{2}}(\mu_2),$$

where M_3 is defined in (5.29) for $\mu_2<\mu_1$ or (5.31) for $\mu_2=\mu_1=\mu$. Moreover, by (5.30), we have

$$\frac{1}{N}S^{\frac{N}{2}} < C_{MP} < \frac{1}{N}S^{\frac{N}{2}}(\mu_1)+\frac{1}{N}S^{\frac{N}{2}}(\mu_2) \leq \frac{2}{N}S^{\frac{N}{2}}.$$

It follows that $\frac{1}{N}S^{\frac{N}{2}}(\mu_2) < \frac{2}{N}S^{\frac{N}{2}}(\mu_1)$. If $\mu_2<\mu_1$, then we have

$$\frac{1+M_3}{2}S^{\frac{N}{2}}(\mu_2) < \frac{2}{N}S^{\frac{N}{2}}(\mu_1) < C_{MP} < \frac{3}{N}S^{\frac{N}{2}}(\mu_1);$$

if $\mu_2=\mu_1=\mu$, we get that

$$\frac{1}{N}S^{\frac{N}{2}}(\mu_1) < \frac{1+M_3}{2}S^{\frac{N}{2}}(\mu_2) < C_{MP} < \frac{2}{N}S^{\frac{N}{2}}(\mu_1).$$

Thus, C_{MP} satisfies (3.10) and (3.11). □

5.4 The case of $N=4, \frac{1}{2} < \big(\frac{1-\mu_1}{1-\mu_2}\big)^{\frac{3}{2}}, 2\big(\frac{S(\mu_2)+S(\mu_1)}{2}\big)^{\frac{N}{2}} > S^{\frac{N}{2}}$.

In this case, if $\mu_2<\mu_1$, M_3' is redefined as

$$M_3' := \frac{1}{2}\Big\{M_1', M_2', 1-\frac{2}{[(1-\mu_1)^{\frac{3}{4}}+(1-\mu_2)^{\frac{3}{4}}]^2}\Big\}, \tag{5.33}$$

where M_1', M_2' are defined in (5.22) and (5.23). Then (5.25) is satisfied and

$$2(1-M_3')\big(\frac{S(\mu_1)+S(\mu_2)}{2}\big)^{\frac{N}{2}} > S^{\frac{N}{2}}. \tag{5.34}$$

If $\mu_2=\mu_1=\mu$, M_3' is redefined as

$$M_3' := \frac{1}{2}-\frac{1}{4(1-4\mu)}, \tag{5.35}$$

then (5.27) and (5.34) are satisfied. Now M_4' is defined by

$$M_4' := \frac{1 - \frac{1}{2}(1 - M_3')^{\frac{2}{N}}}{(\alpha+\beta)(\frac{1}{2})^{\frac{(N-2)(\alpha+\beta-2)}{4}}}. \tag{5.36}$$

Similarly, we have the following result:

Lemma 5.4. *Assume $N = 4, \alpha = \beta = 2, \frac{1}{2} < (\frac{1-\mu_1}{1-\mu_2})^{\frac{3}{2}}$ such that*

$$2\Big(\frac{S(\mu_2) + S(\mu_1)}{2}\Big)^{\frac{N}{2}} > S^{\frac{N}{2}}.$$

Suppose that the weight function $h(x)$ satisfies (H_2) and (H_1'); $\Theta < C_{\alpha,\beta,\mu_1,\mu_2}$ (see (5.1)). Then, if $\Theta \leq \min\{M_4', C_{\alpha,\beta}'\}$, $\overline{\Phi}$ has a mountain pass geometry at the level c satisfying both (3.66) and (3.67).

Proof. It is analogous to the proof of Lemmas 5.2 and 5.3. □

Based on the results of Lemma 3.2 $\sim$ Lemma 5.4, we can obtain the existence of mountain pass solution to problem (1.1).

Proof of Theorem 1.2. Define

$$\begin{aligned} d_1 &:= \min\Big\{0.3S^{\frac{1}{2}}(\mu_1), 10^{-3}\big[(1-4\mu_1)^{\frac{2}{3}} - \big(\frac{1}{2}\big)^{\frac{2}{3}}\big(1-4\mu_2\big)^{\frac{2}{3}}\big], 5\times 10^{-4}\Big\} \\ &= \min\Big\{\frac{3}{10}, \frac{3}{10}S^{\frac{1}{2}}(\mu_1), \frac{\sqrt{2}}{12}, \frac{9}{100}, \frac{9}{100}S^{-\frac{1}{2}}(\mu_1), \\ &\quad 10^{-3}\big[(1-4\mu_1)^{\frac{2}{3}} - \big(\frac{1}{2}\big)^{\frac{2}{3}}\big(1-4\mu_2\big)^{\frac{2}{3}}\big], 5\times 10^{-4}\Big\}. \end{aligned}$$

Then $d_1 < \min\{C_{\alpha,\beta,\mu_1,\mu_2}, M_4, C_{\alpha,\beta}', C_1, C_2\}$. In particular, the assumption (1.14)$\Rightarrow \Theta \leq d_1$.

- Case I: $S^{\frac{N}{2}}(\mu_2) + S^{\frac{N}{2}}(\mu_1) \leq S^{\frac{N}{2}}$.

If $\min\{\alpha, \beta = 2\}$, then $\Theta < d_1$ implies that $\Theta < C_{\alpha,\beta,\mu_1,\mu_2}$. Furthermore,

$$\Theta < d_1 \Rightarrow \Theta < \min\{M_4, C_{\alpha,\beta}'\},$$

where M_4 is defined in (5.12). Then by Lemma 5.1, $\overline{\Phi}$ exhibits a mountain pass geometry and the mountain pass level satisfies (3.10) and (3.11).

- Case II: $2\Big(\frac{S(\mu_1) + S(\mu_2)}{2}\Big)^{\frac{N}{2}} > S^{\frac{N}{2}}$.

For this case, M_4 is defined in (5.32). Analogously, by Lemma 5.3, $\overline{\Phi}$ has a mountain pass geometry with energy level satisfying both (3.10) and (3.11).

For either case I or case II, by the Mountain Pass Theorem, there exists a sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \overline{\mathcal{N}}$ such that

$$\overline{\Phi}(u_n, v_n) \to C_{MP},\ (\overline{\Phi})'\big|_{\overline{\mathcal{N}}}(u_n, v_n) \to 0 \text{ and}$$

$$\overline{\Phi}(u_n, v_n) > \frac{1+M_3}{N} S^{\frac{N}{2}}(\mu_2).$$

Recall that $\Theta < d_1$ implies that $\Theta \leq \min\{C_1, C_2\}$, where C_1, C_2 are defined in (3.8) and (3.9). By Lemma 3.2, $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ admits a subsequence which converges strongly to a critical point (u_0, v_0) of $\overline{\Phi}|_{\overline{\mathcal{N}}}$, which is also a critical point of $\overline{\Phi}$ in $\mathbb{D}$. We observe that $u_0 \geq 0, v_0 \geq 0, u_0 v_0 \not\equiv 0$, hence (u_0, v_0) is a critical point of Φ in $\mathbb{D}$. That is, (u_0, v_0) is a nonnegative mountain pass solution of problem (1.1). □

Proof of Theorem 1.3. Define

$$\begin{aligned} d_2 := &\min\Big\{\frac{\mu_1+\mu_2-(\mu_1+\mu_2)^2}{6}, 10^{-3}\big[(1-4\mu_1)^{\frac{2}{3}} - \big(\frac{1}{2}\big)^{\frac{2}{3}}\big(1-4\mu_2\big)^{\frac{2}{3}}\big],\\ &5\times 10^{-4}, \frac{3}{10}S^{\frac{1}{2}}(\mu_1)\Big\}\\ = &\min\Big\{\frac{\mu_1+\mu_2-(\mu_1+\mu_2)^2}{6}, 10^{-3}\big[(1-4\mu_1)^{\frac{2}{3}} - \big(\frac{1}{2}\big)^{\frac{2}{3}}\big(1-4\mu_2\big)^{\frac{2}{3}}\big],\\ &5\times 10^{-4}, \frac{3}{10}, \frac{3}{10}S^{\frac{1}{2}}(\mu_1), \frac{\sqrt{2}}{12}, \frac{9}{100}, \frac{9}{100}S^{-\frac{1}{2}}(\mu_1)\Big\}. \end{aligned}$$

Then

$$d_2 < \min\Big\{\frac{1-(1-\varepsilon_1)^{\frac{2}{N}}}{2^{\frac{\beta}{2}}\alpha(1-\varepsilon_1)^{\frac{\alpha-2}{2^*}}}, \frac{1-(1-\varepsilon_1)^{\frac{2}{N}}}{2^{\frac{\alpha}{2}}\beta(1-\varepsilon_1)^{\frac{\beta-2}{2^*}}}, C_1, C_2, C_{\alpha,\beta,\mu_1,\mu_2}, M_4, C'_{\alpha,\beta}\Big\},$$

and (1.16)$\Rightarrow \Theta \leq d_2$. Similar to the proof of Theorem 1.2, based on the results of Lemma 5.1 and Lemma 5.3, when $\Theta \leq d_2$, we obtain that $\overline{\Phi}$ has a mountain pass geometry which energy level satisfies both (3.10) and (3.11). By the Mountain Pass Theorem, there exists a sequence $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset \overline{\mathcal{N}}$ such that

$$\overline{\Phi}(u_n, v_n) \to C_{MP},\ (\overline{\Phi})'\big|_{\overline{\mathcal{N}}}(u_n, v_n) \to 0 \text{ and}$$

$$\overline{\Phi}(u_n, v_n) > \frac{1+M_3}{N} S^{\frac{N}{2}}(\mu_2).$$

Notice that

$$\Theta \leq d_2 \Rightarrow \Theta \leq \min\Big\{\frac{1-(1-\varepsilon_1)^{\frac{2}{N}}}{2^{\frac{\beta}{2}}\alpha(1-\varepsilon_1)^{\frac{\alpha-2}{2^*}}}, \frac{1-(1-\varepsilon_1)^{\frac{2}{N}}}{2^{\frac{\alpha}{2}}\beta(1-\varepsilon_1)^{\frac{\beta-2}{2^*}}}, C_1, C_2\Big\},$$

where ε_1 satisfies (3.52) and C_1, C_2 are defined in (3.8) and (3.9). By Lemma 3.3, $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ admits a subsequence which converges strongly to a critical

point (u_0, v_0) of $\overline{\Phi}|_{\overline{\mathcal{N}}}$, which is also a critical point of $\overline{\Phi}$ in $\mathbb{D}$. Also we know that $u_0 \geq 0, v_0 \geq 0, u_0 v_0 \not\equiv 0$, hence (u_0, v_0) is a critical point of Φ in $\mathbb{D}$. That is, (u_0, v_0) is a nonnegative mountain pass solution of the problem (1.1). $\square$

Proof of Theorem 1.4. Define

$$\begin{aligned} d_3 &:= \min\Big\{\frac{2-(1-\mu_1)^{\frac{3}{2}}-(1-\mu_2)^{\frac{3}{2}}}{8}, \frac{2-\sqrt{2}}{4}\Big\} \\ &= \min\Big\{\frac{2-(1-\mu_1)^{\frac{3}{2}}-(1-\mu_2)^{\frac{3}{2}}}{8}, \frac{2-\sqrt{2}}{4}, \frac{1}{4}, \frac{2-\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\Big\}. \end{aligned}$$

Then $d_3 < \min\Big\{\frac{2-(1-\mu_1)^{\frac{3}{2}}-(1-\mu_2)^{\frac{3}{2}}}{8}, \frac{2-\sqrt{2}}{4}, M_4', C'_{\alpha,\beta}, C_{\alpha,\beta,\mu_1,\mu_2}\Big\}$ and (1.19)$\Rightarrow \Theta \leq d_3$. Thus by using Lemma 3.4, Lemma 5.2 and Lemma 5.4, the problem (1.17) has a nontrivial weak solution (u_0, v_0) such that $u_0 \geq 0, v_0 \geq 0, u_0 v_0 \not\equiv 0$. $\square$

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